

FLAT RANK 2 VECTOR BUNDLES ON GENUS 2 CURVES

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ABSTRACT. We study the moduli space of trace-free irreducible rank 2 connections over a curve of genus 2 and the forgetful map towards the moduli space of underlying vector bundles (including unstable bundles), for which we compute a natural Lagrangian rational section. As a particularity of the genus 2 case, connections as above are invariant under the hyperelliptic involution : they descend as rank 2 logarithmic connections over the Riemann sphere. We establish explicit links between the well-known moduli space of the underlying parabolic bundles with the classical approaches by Narasimhan-Ramanan, Tyurin and Bertram. This allow us to explain a certain number of geometric phenomena in the considered moduli spaces such as the classical $(16, 6)$ -configuration of the Kummer surface. We also recover a Poincaré family due to Bolognesi on a degree 2 cover of the Narasimhan-Ramanan moduli space. We explicitly compute the Hitchin integrable system on the moduli space of Higgs bundles and compare the Hitchin Hamiltonians with those found by vanGeemen-Previato. We explicitly describe the isomonodromic foliation in the moduli space of vector bundles with $\mathfrak{sl}_2\mathbb{C}$ -connection over curves of genus 2 and prove the transversality of the induced flow with the locus of unstable bundles.

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INTRODUCTION

Let X be a smooth projective curve of genus 2 over \mathbb{C} . A rank 2 holomorphic connection on X is the data (E, ∇) of a rank 2 vector bundle $E \rightarrow X$ together with a \mathbb{C} -linear map $\nabla : E \rightarrow E \otimes \Omega_X^1$ satisfying the Leibniz rule. The trace $\text{tr}(\nabla)$ defines a holomorphic connection on $\det(E)$; we say that (E, ∇) is trace-free (or a \mathfrak{sl}_2 -connection) when $(\det(E), \text{tr}(\nabla))$ is the trivial connection (\mathcal{O}_X, dz) . From the analytic point of view, (E, ∇) is determined (up to bundle isomorphism) by its monodromy representation, *i.e.* an element of $\text{Hom}(\pi_1(X), \text{SL}_2)/\text{PGL}_2$ (up to conjugacy). The goal of this paper is the explicit construction and study of the moduli stack $\mathfrak{Con}(X)$ of these connections and in particular the forgetful map $(E, \nabla) \mapsto E$ towards the moduli stack $\mathfrak{Bun}(X)$ of vector bundles that can be endowed with connections. Over an open set of the base, the map $\text{bun} : \mathfrak{Con}(X) \rightarrow \mathfrak{Bun}(X)$ is known to be an affine \mathbb{A}^3 -bundle. The former moduli space may be constructed by Geometric Invariant Theory (see [51, 37, 38]) and we get a quasi-projective variety $\text{Con}^{ss}(X)$ whose stable locus $\text{Con}^s(X)$ is open, smooth and parametrizes equivalence classes of irreducible connections. In the strictly semi-stable locus however, several equivalence classes of reducible connections may be identified to the same point.

The moduli space of bundles, even after restriction to the moduli space $\mathfrak{Bun}^{\text{irr}}(X)$ of those bundles admitting an irreducible connection, is non Hausdorff as a topological space, due to the fact that some unstable bundles arise in this way. We can start with the classical moduli space $\text{Bun}^{ss}(X)$ of semi-stable bundles constructed by Narasimhan-Ramanan (see [49]), but we have to investigate how to complete this picture with missing flat unstable bundles.

Hyperelliptic descent. The main tool of our study, elaborated in Section 2, directly follows from the hyperellipticity property of such objects. Denote by $\iota : X \rightarrow X$ the hyperelliptic involution, by $\pi : X \rightarrow \mathbb{P}^1$ the quotient map and by \underline{W} the critical divisor on \mathbb{P}^1 (projection of the 6 Weierstrass points). We can think of $\mathbb{P}^1 = X/\iota$ as an orbifold quotient (see [53]) and any representation $\rho \in \text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2)$ of the orbifold fundamental group, *i.e.* with 2-torsion around points of \underline{W} , can be lifted on X to define an element $\pi^*\rho$ in $\text{Hom}(\pi_1(X), \text{SL}_2)$. As a particularity of the genus 2 case, both moduli spaces of representations have the same dimension 6 and one can check that the map $\text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2) \rightarrow \text{Hom}(\pi_1(X), \text{SL}_2)$ is dominant: any irreducible SL_2 -representation of the fundamental group of X is in the image, is invariant under the hyperelliptic involution ι and can be pushed down to X/ι .

From the point of view of connections, this means that every irreducible connection (E, ∇) on X is invariant by the hyperelliptic involution $\iota : X \rightarrow X$. By pushing forward (E, ∇) to the quotient $X/\iota \simeq \mathbb{P}^1$, we get a rank 4 logarithmic connection that splits into the direct sum $\pi_*(E, \nabla) = (\underline{E}_1, \underline{\nabla}_1) \oplus (\underline{E}_2, \underline{\nabla}_2)$ of two rank 2 connections. Precisely, each \underline{E}_i has degree -3 and $\underline{\nabla}_i : \underline{E}_i \rightarrow \underline{E}_i \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$ is logarithmic with residual eigenvalues 0 and $\frac{1}{2}$ at each pole. Conversely, $\pi^*(\underline{E}_i, \underline{\nabla}_i)$ is a logarithmic connection on X with only apparent singular points: residual eigenvalues are now 0 and 1 at each pole, *i.e.* at each Weierstrass point of the curve. After performing a birational bundle modification (an elementary transformation over each of the 6 Weierstrass points) one can turn it into a holomorphic and trace-free connection on X : we recover the initial connection (E, ∇) . In restriction to the irreducible locus, we deduce a $(2 : 1)$ map $\Phi : \mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X)$ where $\mathfrak{Con}(X/\iota)$ denotes the moduli space of logarithmic connections like above. Moduli spaces of logarithmic connections on \mathbb{P}^1 have been widely studied by many authors. Note that the idea of descent to \mathbb{P}^1 for studying sheaves on hyperelliptic curves already appears in work of S. Ramanan and his student U. Bhosle (see for example [57] and [8]).

One can associate to a connection $(\underline{E}, \underline{\nabla}) \in \mathfrak{Con}(X/\iota)$ a parabolic structure \underline{p} on \underline{E} consisting of the data of the residual eigenspace $p_j \subset \underline{E}|_{w_j}$ associated to the $\frac{1}{2}$ -eigenvalue for each pole w_j in the support of \underline{W} . Denote by $\mathfrak{Bun}(X/\iota)$ the moduli space of such parabolic bundles $(\underline{E}, \underline{p})$, *i.e.* defined by a logarithmic connection $(\underline{E}, \underline{\nabla}) \in \mathfrak{Con}(X/\iota)$. In fact, the descending procedure described above can already be constructed at the level of bundles (see [9]) and we can construct a $(2 : 1)$ map $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$ making the following diagram commutative:

$$(1) \quad \begin{array}{ccc} \mathfrak{Con}(X/\iota) & \xrightarrow[\Phi]{2:1} & \mathfrak{Con}(X) \\ \text{bun} \downarrow & & \downarrow \text{bun} \\ \mathfrak{Bun}(X/\iota) & \xrightarrow[\phi]{2:1} & \mathfrak{Bun}(X) \end{array}$$

Vertical arrows are locally trivial affine \mathbb{A}^3 -bundles in restriction to a large open set of the bases.

Narasimhan-Ramanan moduli space. Having this picture at hand, we study in Section 3 the structure of $\mathfrak{Bun}(X)$, partly surveying Narasimhan-Ramanan's classical work [49]. They construct a quotient map

$$\mathrm{NR} : \mathrm{Bun}^{ss}(X) \rightarrow \mathbb{P}_{\mathrm{NR}}^3 := |2\Theta|$$

defined on the open set $\mathrm{Bun}^{ss}(X) \subset \mathfrak{Bun}(X)$ of semi-stable bundles onto the 3-dimensional linear system generated by twice the Θ -divisor on $\mathrm{Pic}^1(X)$. This map is one-to-one in restriction to the open set $\mathrm{Bun}^s(X)$ of stable bundles; it however identifies some strictly semi-stable bundles, as usually does GIT theory to get a Hausdorff quotient. Precisely, the Kummer surface $\mathrm{Kum}(X) = \mathrm{Jac}(X)/_{\pm 1}$ naturally parametrizes the set of decomposable semi-stable bundles, and the classifying map NR provides an embedding $\mathrm{Kum}(X) \hookrightarrow \mathbb{P}_{\mathrm{NR}}^3$ as a quartic surface with 16 nodes. The open set of stable bundles is therefore parametrized by the complement $\mathbb{P}_{\mathrm{NR}}^3 \setminus \mathrm{Kum}(X)$. Over a smooth point of $\mathrm{Kum}(X)$, the fiber of NR consists in 3 isomorphism classes of semi-stable bundles, namely a decomposable one $L_0 \oplus L_0^{-1}$ and the two non trivial extensions between L_0 and L_0^{-1} . The latter ones, which we call affine bundles, are precisely the bundles occurring in $\mathfrak{Bun}(X) \setminus \mathfrak{Bun}^{\mathrm{irr}}(X)$, where $\mathfrak{Bun}^{\mathrm{irr}}(X)$ denotes the moduli space of rank 2 bundles over X that can be endowed with an irreducible trace-free connection. Over each singular point of $\mathrm{Kum}(X)$, the fiber of NR consists in a decomposable bundle E_τ (a twist of the trivial bundle by a 2-torsion point τ of $\mathrm{Jac}(X)$) and the (rational) one-parameter family of non trivial extensions of τ by itself. The latter ones we call (twists of) unipotent bundles; each of them is arbitrarily close to E_τ in $\mathfrak{Bun}(X)$. To complete this classical picture, we have to add flat unstable bundles: by Weil's criterion, these are exactly the unique non-trivial extensions $\vartheta \rightarrow E_\vartheta \rightarrow \vartheta^{-1}$ where $\vartheta \in \mathrm{Pic}^1(X)$ runs over the 16 theta-characteristics $\vartheta^2 = K_X$. We call them Gunning bundles in reference to [29]: connections defining a projective PGL_2 -structure on X (an oper in the sense of [5], see also [6]) are defined on these very special bundles E_ϑ , including the uniformization equation for X . These bundles occur as non Hausdorff points of $\mathfrak{Bun}(X)$: the bundles arbitrarily close to E_ϑ are precisely semi-stable extensions of the form $\vartheta^{-1} \rightarrow E \rightarrow \vartheta$. They are sent onto a plane $\Pi_\vartheta \subset \mathbb{P}_{\mathrm{NR}}^3$ by the Narasimhan-Ramanan classifying map. We call them Gunning planes: they are precisely the 16 planes involved in the classical (16,6)-configuration of Kummer surfaces (see [34, 26]). As far as we know, these planes have had no modular interpretation so far. We supplement this geometric study with explicit computations of Narasimhan-Ramanan coordinates, together with the equation of $\mathrm{Kum}(X)$, as well as the 16-order symmetry group. These computations are done for the genus 2 curve defined by an affine equation $y^2 = x(x-1)(x-r)(x-s)(x-t)$ as functions of the free parameters (r, s, t) .

The branching cover $\phi : \mathfrak{Bun}(X/\iota) \xrightarrow{2:1} \mathfrak{Bun}(X)$. In Section 5, we provide a full description of this map which is a double cover of $\mathfrak{Bun}^{\mathrm{irr}}(X)$ branching over the locus of decomposable bundles, including the trivial bundle and its 15 twists. The 16 latter bundles lift as 16 decomposable parabolic bundles. If we restrict ourselves to the complement of these very special bundles, we can follow the previous work of [2, 41]: the moduli space $\mathfrak{Bun}^{\mathrm{ind}}(X/\iota)$ of indecomposable parabolic bundles can be constructed by patching together GIT quotients $\mathrm{Bun}_\mu^{ss}(X/\iota)$ of μ -semi-stable parabolic bundles for a finite number of weights $\mu \in [0, 1]^6$. These moduli spaces are smooth projective manifolds and they are patched together along Zariski open subsets, giving $\mathfrak{Bun}^{\mathrm{ind}}(X/\iota)$ the structure of a smooth non separated scheme. In the present work, we mainly study

a one-parameter family of weights, namely the diagonal family $\boldsymbol{\mu} = (\mu, \mu, \mu, \mu, \mu, \mu)$. For $\mu = \frac{1}{2}$, the restriction map $\phi : \text{Bun}_{\frac{1}{2}}^{ss}(X/\iota) \rightarrow \mathbb{P}_{\text{NR}}^3$ is exactly the 2-fold cover of \mathbb{P}_{NR}^3 ramifying over the Kummer surface $\text{Kum}(X)$. The space $\text{Bun}_{\boldsymbol{\mu}}^{ss}(X/\iota)$ is singular for this special value $\mu = \frac{1}{2}$. We thoroughly study the chart given by any $\frac{1}{6} < \mu < \frac{1}{4}$ which is a 3-dimensional projective space, that we will denote $\mathbb{P}_{\mathbf{b}}^3$: it is naturally isomorphic to the space of extensions studied by Bertram and Bolognesi [7, 14, 15]. The Narasimhan-Ramanan classifying map $\phi : \mathbb{P}_{\mathbf{b}}^3 \dashrightarrow \mathbb{P}_{\text{NR}}^3$ is rational and also related to the classical geometry of Kummer surfaces. There is no universal bundle for the Narasimhan-Ramanan moduli space \mathbb{P}_{NR}^3 , but there is one for the 2-fold cover $\mathbb{P}_{\mathbf{b}}^3$. This universal bundle, due to Bolognesi [15] is explicitly constructed in Section 4.3 from the Tyurin point of view.

We establish a complete dictionary between special (in the sense of non stable) bundles E in $\mathfrak{Bun}(X)$ (listed in Section 3) and special parabolic bundles $(\underline{E}, \underline{p})$ in $\mathfrak{Bun}(X/\iota)$ allowing us to describe the geometry of the non separated singular schemes $\mathfrak{Bun}(X/\iota)$ and $\mathfrak{Bun}^{\text{irr}}(X)$.

Anticanonical subbundles and Tyurin parameters. In order to establish this dictionary, we study in Section 4 the space of sheaf inclusions of the form $\mathcal{O}_X(-K_X) \hookrightarrow E$ for each type of bundle E . This is a 2-dimensional vector space for a generic vector bundle E and defines a 1-parameter family of line subbundles. Only two of these anticanonical subbundles are invariant under the hyperelliptic involution. In the generic case, the fibres over the Weierstrass points of these two subbundles define precisely the two possible parabolic structures \underline{p} and \underline{p}' on E that arise in the context of hyperelliptic descent. This allows us to relate our moduli space $\mathfrak{Bun}(X/\iota)$ to the space of ι -invariant extensions $-K_X \rightarrow E \rightarrow K_X$ studied by Bertram and Bolognesi: their moduli space coincides with our chart $\mathbb{P}_{\mathbf{b}}^3$.

On the other hand, anticanonical morphisms provide, for a generic bundle E , a birational morphism $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X) \rightarrow E$, or after tensoring by $\mathcal{O}_X(K_X)$, a birational and minimal trivialisation $E_0 \rightarrow E$. Precisely, this birational bundle map consists in 4 elementary transformations for a parabolic structure on the trivial bundle E_0 supported by a divisor belonging to the linear system $|2K_X|$. The moduli space of such parabolic structures is a birational model for $\mathfrak{Bun}(X)$ (from which we easily deduce the rationality of this moduli stack).

We provide the explicit change of coordinates between the Tyurin parameters and the other previous parameters.

Higgs bundles and the Hitchin fibration. Section 6 contains some applications of our previous study of diagram (1) to the space of Higgs bundles \mathfrak{Higgs} over X , respectively X/ι , which can be interpreted as the homogeneous part of the affine bundle $\mathfrak{Con} \rightarrow \mathfrak{Bun}$. We provide an explicit universal Higgs bundle for $\mathfrak{Higgs}(X/\iota)$ and we compute the Hitchin Hamiltonians for the Hitchin system on $\mathfrak{Higgs}(X/\iota)$. Using the natural identification with the cotangent bundle $T^*\mathfrak{Bun}(X/\iota)$ together with the double cover $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$, we derive the explicit Hitchin map $\mathfrak{Higgs}(X) \rightarrow H^0(X, 2K_X) ; (E, \Theta) \mapsto \det(\Theta)$ in a very direct way in Section 6.2. This allows us to relate the six Hamiltonians described by G. van Geemen and E. Previato in [23] to the three Hamiltonian coefficients of the Hitchin map.

The geometry of $\mathfrak{Con}(X)$. The computations of the Tyurin parameters in Section 4.3 and their relation to the so-called apparent map on \mathfrak{Con} defined in Section 7.2 allow us to construct an explicit rational section $\mathfrak{Bun}(X) \dashrightarrow \mathfrak{Con}(X)$ which is regular over

the stable open subset of $\mathfrak{Bun}(X)$, and is, moreover, Lagrangian (see Section 7.3). In other words, over the stable open set, the Lagrangian fiber-bundle $\mathfrak{Con}(X) \rightarrow \mathfrak{Bun}(X)$ is isomorphic to the cotangent bundle $T^*\mathfrak{Bun}(X)$ (i.e. $T^*P_{\text{NR}}^3$) as a symplectic manifold. Together with a natural basis of the space of Higgs bundles over X we thereby obtain a universal connection parametrizing an affine chart of $\mathfrak{Con}(X)$.

Isomonodromic deformations. On the moduli stack \mathcal{M} of triples (X, E, ∇) , where X is a genus two curve, and $(E, \nabla) \in \mathfrak{Con}^*(X)$ a ι -invariant but non trivial \mathfrak{sl}_2 -connection on X , isomonodromic deformations form the leaves of a 3-dimensional holomorphic foliation, the isomonodromy foliation. It is locally defined by the fibers of the analytic Riemann-Hilbert map, which to a connection associate its monodromy representation. Our double-cover construction $\Phi : \mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X)$ is compatible with isomonodromic deformations when we let the complex structure of X vary. Therefore, isomonodromic deformation equations for holomorphic SL_2 -connections on X reduce to a Garnier system.

Hence in the moduli stack \mathcal{M} , we can explicitly describe the isomonodromy foliation \mathcal{F}_{iso} as well as the locus of special bundles, for example the locus $\Sigma \subset \mathcal{M}$ of connections on Gunning bundles. We show that the isomonodromy foliation is transverse to the locus of Gunning bundles by direct computation in Theorem 8.1. As a corollary, we obtain a new proof of a result of Hejhal [30], stating that the monodromy map from the space of projective structures on the genus two curves to the space of SL_2 -representations of the fundamental group is a local diffeomorphism.

1. PRELIMINARIES ON CONNECTIONS

In this section, we introduce the objects and methods related to the notion of connection relevant for this paper, such as parabolic logarithmic connections and their elementary transformations. More detailed introductions can be found for example in [55], [27] and [25].

1.1. Logarithmic connections. Let X be a smooth projective curve over \mathbb{C} and $E \rightarrow X$ be a rank r vector bundle. Let D be a reduced effective divisor on X . Note that in general, we make no difference in notation between a reduced effective divisor and its support, as well as between the total space of a vector bundle and its locally free sheaf of holomorphic sections. A *logarithmic connection* on E with polar divisor D is a \mathbb{C} -linear map

$$\nabla : E \rightarrow E \otimes \Omega_X^1(D)$$

satisfying the Leibniz rule

$$\nabla(f \cdot s) = df \otimes s + f \cdot \nabla(s)$$

for any local section s of E and function f on X . Locally, for a trivialization of E , the connection writes $\nabla = d_X + A$ where $d_X : \mathcal{O}_X \rightarrow \Omega_X^1$ is the differential operator on X and A is a $r \times r$ matrix with coefficients in $\Omega_X^1(D)$, thus 1-forms having at most simple poles located along D . The *true polar divisor*, i.e. the singular set of such a logarithmic connection ∇ is a subset of D . Depending on the context, we may assume them to be equal. At each pole $x_0 \in D$, the residual matrix intrinsically defines an endomorphism of the fiber E_{x_0} that we denote $\text{Res}_{x_0} \nabla$. *Residual eigenvalues* and *residual eigenspaces* in E_{x_0} hence are well-defined.

1.2. Twists and trace. As before, let E be a rank r vector bundle endowed with a logarithmic connection ∇ on a curve X . The connection ∇ induces a logarithmic connection $\text{tr}(\nabla)$ on the determinant line bundle $\det(E)$ over X with

$$\text{Res}_{x_0} \text{tr}(\nabla) = \text{tr}(\text{Res}_{x_0} \nabla)$$

for each $x_0 \in D$. By the residue theorem, the sum of residues of a global meromorphic 1-form on X is zero. We thereby obtain *Fuchs' relation*:

$$(2) \quad \deg(E) + \sum_{x_0 \in D} \text{tr}(\text{Res}_{x_0} \nabla) = 0.$$

We can define the *twist* of the connection (E, ∇) by a rank 1 meromorphic connection (L, ζ) as the rank r connection (E', ∇') with

$$(E', \nabla') = (E, \nabla) \otimes (L, \zeta) := (E \otimes L, \nabla \otimes \text{id}_L + \text{id}_E \otimes \zeta).$$

We have

$$\det(E') = \det(E) \otimes L^{\otimes r} \quad \text{and} \quad \text{tr}(\nabla') = \text{tr}(\nabla) \otimes \zeta^{\otimes r}.$$

If $L \rightarrow X$ is a line bundle such that $L^{\otimes r} \simeq \mathcal{O}_X$, then there is a unique (holomorphic) connection ∇_L on L such that the connection $\nabla_L^{\otimes r}$ is the trivial connection on $L^{\otimes r} \simeq \mathcal{O}_X$. The twist by such a r -torsion connection has no effect on the trace: modulo isomorphism, we have $\det(E') = \det(E)$ and $\text{tr}(\nabla') = \text{tr}(\nabla)$.

1.3. Projective connections and Riccati foliations. From now on, let us assume the rank to be $r = 2$. After projectivizing the bundle E , we get a \mathbb{P}^1 -bundle $\mathbb{P}E$ over X whose total space is a ruled surface S . Since ∇ is \mathbb{C} -linear, it defines a projective connection $\mathbb{P}\nabla$ on $\mathbb{P}E$ and the graphs of horizontal sections define a foliation by curves \mathcal{F} on the ruled surface S . The foliation \mathcal{F} is transversal to a generic member of the ruling $S \rightarrow X$ and is thus a Riccati foliation (see [16], chapter 4). If the connection locally writes

$$\nabla : \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto d \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix},$$

then in the corresponding trivialization $(z_1 : z_2) = (1 : z)$ of the ruling, the foliation is defined by the (pfaffian) Riccati equation

$$dz - \beta z^2 + (\delta - \alpha)z + \gamma = 0.$$

Tangencies between \mathcal{F} and the ruling are concentrated on fibers over the (true) polar divisor D of ∇ . These singular fibers are totally \mathcal{F} -invariant. According to the number of residual eigendirections of ∇ , the restriction of \mathcal{F} to such a fibre is the union of a leaf and 1 or 2 points.

Any two connections (E, ∇) and (E', ∇') on X define the same Riccati foliation if, and only if, $(E', \nabla') = (E, \nabla) \otimes (L, \zeta)$ for a rank 1 connection (L, ζ) . Conversely, a Riccati foliation (S, \mathcal{F}) is always the projectivization of a connection (E, ∇) : once we have chosen a lift E of S and a rank 1 connection ζ on $\det(E)$, there is a unique connection ∇ on E such that $\text{trace}(E) = \zeta$ and $\mathbb{P}\nabla = \mathcal{F}$.

1.4. Parabolic structures. A *parabolic structure* on E supported by a reduced divisor $D = x_1 + \dots + x_n$ on X is the data $\mathbf{p} = (p_1, \dots, p_n)$ of a 1-dimensional subspace $p_i \in E_{x_i}$ for each $x_i \in D$. A *parabolic connection* is the data (E, ∇, \mathbf{p}) of a logarithmic connection (E, ∇) with polar divisor D and a parabolic structure \mathbf{p} supported by D such that, at each pole $x_i \in D$, the parabolic direction p_i is an eigendirection of the residual endomorphism $\text{Res}_{x_i} \nabla$. For the corresponding Riccati foliation, \mathbf{p} is the data, on the ruled surface S , of a singular point of the foliation \mathcal{F} for each fiber over D .

Remark 1.1. *Note that our definition is non standard here: in the literature, a parabolic structure on E is usually defined as the data \mathbf{p} (a quasi-parabolic structure) together with a collection of weights $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n) \in \mathbb{R}^n$.*

1.5. Elementary transformations. Let (E, p) be a parabolic bundle on X supported by a single point $x_0 \in X$. Consider the vector bundle E^- defined by the subsheaf of those sections s of E such that $s(x_0) \in p$. A natural parabolic direction on E^- is defined by those sections of E which are vanishing at x_0 (and thus belong to E^-). If x is a local coordinate at x_0 and E is generated near x_0 by $\langle e_1, e_2 \rangle$ with $e_1(x_0) \in p$, then E^- is locally generated by $\langle e_1, e'_2 \rangle$ with $e'_2 := xe_2$ and we define $p^- \subset E^-|_{x_0}$ to be $\mathbb{C}e'_2(x_0)$. By identifying the sections of E and E^- outside x_0 , we obtain a natural birational morphism (see also [42])

$$\text{elm}_{x_0}^- : E \dashrightarrow E^-.$$

In a similar way, we define the parabolic bundle (E^+, p^+) by the sheaf of those meromorphic sections of E having (at most) a single pole at x_0 , whose residual part is an element of p . The parabolic p^+ then is defined by

$$p^+ := \{s(x_0) \mid s \text{ is a holomorphic section of } E \text{ near } x_0\}.$$

In other words, E^+ is generated by $\langle e'_1, e_2 \rangle$ with $e'_1 := \frac{1}{x}e_1$ and $p^+ \subset E^+|_{x_0}$ defined by $\mathbb{C}e_2$. The natural morphism

$$\text{elm}_{x_0}^+ : E \dashrightarrow E^+$$

is now regular, but fails to be an isomorphism at x_0 .

These *elementary transformations* satisfy the following properties:

- $\det(E^\pm) = \det(E) \otimes \mathcal{O}_X(\pm[x_0])$,
- $\text{elm}_{x_0}^+ \circ \text{elm}_{x_0}^- = \text{id}_{(E, p)}$ and $\text{elm}_{x_0}^- \circ \text{elm}_{x_0}^+ = \text{id}_{(E, p)}$,
- $\text{elm}_{x_0}^+ = \mathcal{O}_X([x_0]) \otimes \text{elm}_{x_0}^-$.

In particular, positive and negative elementary transformations coincide for a projective parabolic bundle $(\mathbb{P}E, p)$. They consist, for the ruled surface S , in composing the blowing-up of p with the contraction of the strict transform of the fiber [25]. This latter contraction gives the new parabolic p^\pm . Elementary transformations on projective parabolic bundles are clearly involutive.

More generally, given a parabolic bundle (E, \mathbf{p}) with support D , we define the elementary transformations elm_D^\pm as the composition of the (commuting) single elementary transformations over all points of D . We define $\text{elm}_{D_0}^\pm$ for any subdivisor $D_0 \subset D$ in the obvious way.

Given a parabolic connection (E, ∇, \mathbf{p}) with support D , the elementary transformations elm_D^\pm yield new parabolic connections $(E^\pm, \nabla^\pm, \mathbf{p}^\pm)$. In fact, the compatibility condition between \mathbf{p} and the residual eigenspaces of ∇ insures that ∇^\pm is still logarithmic. The monodromy is obviously left unchanged, but the residual eigenvalues are shifted as

follows: if λ_1 and λ_2 denote the residual eigenvalues of ∇ at x_0 , with p contained in the λ_1 -eigenspace, then

- ∇^+ has eigenvalues $(\lambda_1^+, \lambda_2^+) := (\lambda_1 - 1, \lambda_2)$,
- ∇^- has eigenvalues $(\lambda_1^-, \lambda_2^-) := (\lambda_1, \lambda_2 + 1)$,

and p^\pm is now defined by the λ_2^\pm -eigenspace.

Finally, if the parabolic connections (E, ∇, \mathbf{p}) and $(\tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}})$ are isomorphic, then one can easily check that $(E^\pm, \nabla^\pm, \mathbf{p}^\pm)$ and $(\tilde{E}^\pm, \tilde{\nabla}^\pm, \tilde{\mathbf{p}}^\pm)$ are also isomorphic. This will allow us to define elementary transformations elm_D^\pm on moduli spaces of parabolic connections.

1.6. Stability and moduli spaces. Given a collection $\boldsymbol{\mu} = (\mu_1, \dots, \mu_n)$ of weights $\mu_i \in [0, 1]$ attached to p_i , we define the *parabolic degree* with respect to $\boldsymbol{\mu}$ of a line subbundle $L \hookrightarrow E$ as

$$\deg_{\boldsymbol{\mu}}^{\text{par}}(L) := \deg(L) + \sum_{p_i \subset L} \mu_i$$

(where the summation is taken over those parabolics p_i contained in the total space of $L \subset E$). Setting

$$\deg_{\boldsymbol{\mu}}^{\text{par}}(E) := \deg(E) + \sum_{i=1}^n \mu_i$$

(where the summation is taken over all parabolics), we define the *stability index* of L by

$$\text{ind}_{\boldsymbol{\mu}}(L) := \deg_{\boldsymbol{\mu}}^{\text{par}}(E) - 2 \deg_{\boldsymbol{\mu}}^{\text{par}}(L).$$

The parabolic bundle (E, \mathbf{p}) is called *semi-stable* (resp. *stable*) with respect to $\boldsymbol{\mu}$ if

$$\text{ind}_{\boldsymbol{\mu}}(L) \geq 0 \text{ (resp. } > 0 \text{) for each line subbundle } L \subset E.$$

For vanishing weights $\mu_1 = \dots = \mu_n = 0$, we get the usual definition of (semi-)stability of vector bundles. We say a bundle is *strictly semi-stable* if it is semi-stable but not stable. A bundle is called *unstable* if it is not semi-stable.

Semi-stable parabolic bundles admit a coarse moduli space $\text{Bun}_{\boldsymbol{\mu}}^{ss}$ which is a normal projective variety; the stable locus $\text{Bun}_{\boldsymbol{\mu}}^s$ is smooth (see [46]). Note that tensoring by a line bundle does not affect the stability index. In fact, if S denotes again the ruled surface defined by $\mathbb{P}E$, line bundles $L \hookrightarrow E$ are in one to one correspondence with sections $\sigma : X \rightarrow S$, and for vanishing weights, $\text{ind}_{\boldsymbol{\mu}}(L)$ is precisely the self-intersection number of the curve $C := \sigma(X) \subset S$ (see also [43]). For general weights, we have

$$\text{ind}_{\boldsymbol{\mu}}(L) = \#(C \cdot C) + \sum_{p_i \notin C} \mu_i - \sum_{p_i \in C} \mu_i.$$

For weighted parabolic bundles $(E, \mathbf{p}, \boldsymbol{\mu})$, it is natural to extend the definition of elementary transformations as follows. Given a subdivisor $D_0 \subset D$, define

$$\text{elm}_{D_0}^\pm : (E, \mathbf{p}, \boldsymbol{\mu}) \dashrightarrow (E', \mathbf{p}', \boldsymbol{\mu}')$$

by setting

$$\mu'_i = \begin{cases} 1 - \mu_i & \text{if } p_i \in D_0, \\ \mu_i & \text{if } p_i \notin D_0. \end{cases}$$

When $L' \hookrightarrow E'$ denotes the strict transform of L , we can easily check that

$$\text{ind}_{\boldsymbol{\mu}'}(L') = \text{ind}_{\boldsymbol{\mu}}(L).$$

Therefore, $\text{elm}_{D_0}^\pm$ acts as an isomorphism between the moduli spaces Bun_μ^{ss} and $\text{Bun}_{\mu'}^{ss}$ (resp. Bun_μ^s and $\text{Bun}_{\mu'}^s$). A parabolic connection $(E, \nabla, \underline{p})$ is said to be *semi-stable* (resp. *stable*) with respect to μ if

$$\text{ind}_\mu(L) \geq 0 \text{ (resp. } > 0) \text{ for all } \nabla\text{-invariant line subbundles } L \subset E.$$

In particular, irreducible connections are stable for any weight $\mu \in [0, 1]^n$. Semi-stable parabolic connections admit a coarse moduli space Con_μ^{ss} which is a normal quasi-projective variety; the stable locus Con_μ^s is smooth (see [51]).

2. HYPERELLIPTIC CORRESPONDENCE

Let X be the smooth complex projective curve given in an affine chart of $\mathbb{P}^1 \times \mathbb{P}^1$ by

$$y^2 = x(x-1)(x-r)(x-s)(x-t).$$

Denote its hyperelliptic involution, defined in the above chart by $(x, y) \mapsto (x, -y)$, by $\iota : X \rightarrow X$ and denote its hyperelliptic cover, defined in the above chart by $(x, y) \mapsto x$, by $\pi : X \rightarrow \mathbb{P}^1$. Denote by $\underline{W} = \{0, 1, r, s, t, \infty\}$ the critical divisor on \mathbb{P}^1 and by $W = \{w_0, w_1, w_r, w_s, w_t, w_\infty\}$ the *Weierstrass divisor* on X , *i.e.* the branching divisor with respect to π .

Consider a rank 2 vector bundle $\underline{E} \rightarrow \mathbb{P}^1$ of degree -3 , endowed with a logarithmic connection $\underline{\nabla} : \underline{E} \rightarrow \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$ having residual eigenvalues 0 and $\frac{1}{2}$ at each pole. We fix the parabolic structure \underline{p} attached to the $\frac{1}{2}$ -eigenspaces over \underline{W} . After lifting the parabolic connection $(\underline{E}, \underline{\nabla}, \underline{p})$ via $\pi : X \rightarrow \mathbb{P}^1$, we get a parabolic connection on X

$$(\tilde{E} \rightarrow X, \tilde{\nabla}, \tilde{p}) = \pi^*(\underline{E} \rightarrow \mathbb{P}^1, \underline{\nabla}, \underline{p}).$$

We have $\det(\tilde{E}) \simeq \mathcal{O}_X(-3K_X)$ and the residual eigenvalues of the connection $\tilde{\nabla} : \tilde{E} \rightarrow \tilde{E} \otimes \Omega_X^1(W)$ are 0 and 1 at each pole, with parabolic structure \tilde{p} defined by the 1-eigenspaces. After applying elementary transformations directed by \tilde{p} , we get a new parabolic connection:

$$\text{elm}_W^+ : (\tilde{E}, \tilde{\nabla}, \tilde{p}) \dashrightarrow (E, \nabla, p)$$

which is now holomorphic and trace-free.

Recall from the introduction that we denote by $\mathfrak{Con}(X/\iota)$ the moduli space of logarithmic rank 2 connections on \mathbb{P}^1 with residual eigenvalues 0 and $\frac{1}{2}$ at each pole in \underline{W} , and we denote by $\mathfrak{Con}(X)$ the moduli space of trace-free holomorphic rank 2 connections on X . Since to every element $(\underline{E}, \underline{\nabla})$ of $\mathfrak{Con}(X/\iota)$, the parabolic structure \underline{p} is intrinsically defined as above, we have just defined a map

$$\Phi : \begin{cases} \mathfrak{Con}(X/\iota) & \rightarrow \mathfrak{Con}(X) \\ (\underline{E}, \underline{\nabla}, \underline{p}) & \mapsto (E, \nabla). \end{cases}$$

Roughly counting dimensions, we see that both spaces of connections have same dimension 6 up to bundle isomorphisms. We may expect to obtain most of all holomorphic and trace-free rank 2 connections on X by this construction. This turns out to be true and will be proved along Section 2.1. In particular, any *irreducible* holomorphic and trace-free rank 2 connection (E, ∇) on X can be obtained like above. Note that the stability of E is a sufficient condition for the irreducibility of ∇ .

2.1. Topological considerations. By the Riemann-Hilbert correspondence, the two moduli spaces of connections considered above are in one-to-one correspondence with moduli spaces of representations. Let us start with $\mathfrak{Con}(X)$ which is easier. The monodromy of a trace-free holomorphic rank 2 connection (E, ∇) on X gives rise to a monodromy representation, namely a homomorphism $\rho : \pi_1(X, w) \rightarrow \mathrm{SL}_2$. In fact, this depends on the choice of a basis on the fiber E_w . Another choice will give the conjugate representation $M\rho M^{-1}$ for some $M \in \mathrm{GL}_2$. The class $[\rho] \in \mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2) / \mathrm{PGL}_2$ however is well-defined by (E, ∇) . Conversely, the monodromy $[\rho]$ characterizes the connection (E, ∇) on X modulo isomorphism, which yields a bijective correspondence

$$\mathrm{RH} : \mathfrak{Con}(X) \xrightarrow{\sim} \mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2) / \mathrm{PGL}_2$$

which turns out to be complex analytic where it makes sense, *i.e.* on the smooth part. Yet this map is highly transcendental, since we have to integrate a differential equation to compute the monodromy. Note that the space of representations only depends on the topology of X , not on the complex and algebraic structure.

In a similar way, parabolic connections in $\mathfrak{Con}(X/\iota)$ are in one-to-one correspondence with faithful representations $\rho : \pi_1^{\mathrm{orb}}(X/\iota) \rightarrow \mathrm{GL}_2$ of the orbifold fundamental group (killing squares of simple loops around punctures, see the proof of theorem 2.1 below). Thinking of $\mathbb{P}^1 = X/\iota$ as the orbifold quotient of X by the hyperelliptic involution, these representations can also be seen as representations

$$\rho : \pi_1(\mathbb{P}^1 \setminus \underline{W}, x) \rightarrow \mathrm{GL}_2$$

with 2-torsion monodromy around the punctures, having eigenvalues 1 and -1 .

If $x = \pi(w)$, the branching cover $\pi : X \rightarrow X/\iota$ induces a monomorphism

$$\pi_* : \pi_1(X, w) \hookrightarrow \pi_1^{\mathrm{orb}}(X/\iota, x),$$

whose image consists of words of even length in the alphabet of a system of simple generators of $\pi_1^{\mathrm{orb}}(X/\iota, x)$. This allows to associate, to any representation $\rho : \pi_1^{\mathrm{orb}}(X/\iota, x) \rightarrow \mathrm{GL}_2$ as above, a representation $\rho \circ \pi_* : \pi_1(X, w) \rightarrow \mathrm{SL}_2$. We have thereby defined a map Φ^{top} between corresponding representation spaces, which makes the following diagram commutative

$$(3) \quad \begin{array}{ccc} \mathfrak{Con}(X/\iota) & \xrightarrow[\sim]{\mathrm{RH}} & \mathrm{Hom}(\pi_1^{\mathrm{orb}}(X/\iota, x), \mathrm{GL}_2) / \mathrm{PGL}_2 \\ \downarrow \Phi & & \downarrow \Phi^{\mathrm{top}} \\ \mathfrak{Con}(X) & \xrightarrow[\sim]{\mathrm{RH}} & \mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2) / \mathrm{PGL}_2. \end{array}$$

We now want to describe the map Φ^{top} . The quotient $\pi_1^{\mathrm{orb}}(X/\iota, x) / \pi_*(\pi_1(X, w)) \simeq \mathbb{Z}_2$ acts (by conjugacy) as outer automorphisms of $\pi_1(X, w)$. It coincides with the outer action of the hyperelliptic involution ι .

Theorem 2.1. *Given a representation $[\rho] \in \mathrm{Hom}(\pi_1(X), \mathrm{SL}_2) / \mathrm{PGL}_2$, the following properties are equivalent:*

- (a) $[\rho]$ is either irreducible or abelian;
- (b) $[\rho]$ is ι -invariant;
- (c) $[\rho]$ is in the image of Φ^{top} .

If these properties are satisfied, then $[\rho]$ has 1 or 2 preimages under Φ^{top} , depending on whether it is diagonal or not.

Proof. We start making explicit the monomorphism π_* and the involution ι . Let $x \in \mathbb{P}^1 \setminus \underline{W}$ and $w \in X$ one of the two preimages. Choose simple loops around the punctures to generate the orbifold fundamental group of $\mathbb{P}^1 \setminus \underline{W}$ with the standard representation

$$\pi_1^{\text{orb}}(X/\iota, x) = \left\langle \gamma_0, \gamma_1, \gamma_r, \gamma_s, \gamma_t, \gamma_\infty \mid \begin{array}{l} \gamma_0^2 = \gamma_1^2 = \gamma_r^2 = \gamma_s^2 = \gamma_t^2 = \gamma_\infty^2 = 1 \\ \text{and } \gamma_0\gamma_1\gamma_r\gamma_s\gamma_t\gamma_\infty = 1 \end{array} \right\rangle.$$

Even words in these generators can be lifted as loops based in w on X , generating the ordinary fundamental group of X . Using the relations, we see that $\pi_1(X, w)$ is actually generated by the following pairs

$$\begin{cases} \alpha_1 := \gamma_0\gamma_1 \\ \beta_1 := \gamma_r\gamma_1 \end{cases} \quad \begin{cases} \alpha_2 := \gamma_s\gamma_t \\ \beta_2 := \gamma_\infty\gamma_t \end{cases}$$

and they provide the standard presentation

$$(4) \quad \pi_1(X, w) = \langle \alpha_1, \beta_1, \alpha_2, \beta_2 \mid [\alpha_1, \beta_1][\alpha_2, \beta_2] = 1 \rangle,$$

where $[\alpha_i, \beta_i] = \alpha_i\beta_i\alpha_i^{-1}\beta_i^{-1}$ denotes the commutator. In other words, the monomorphism π_* is defined by $\alpha_1 \mapsto \gamma_0\gamma_1$ *et cetera* (see Figure 1).

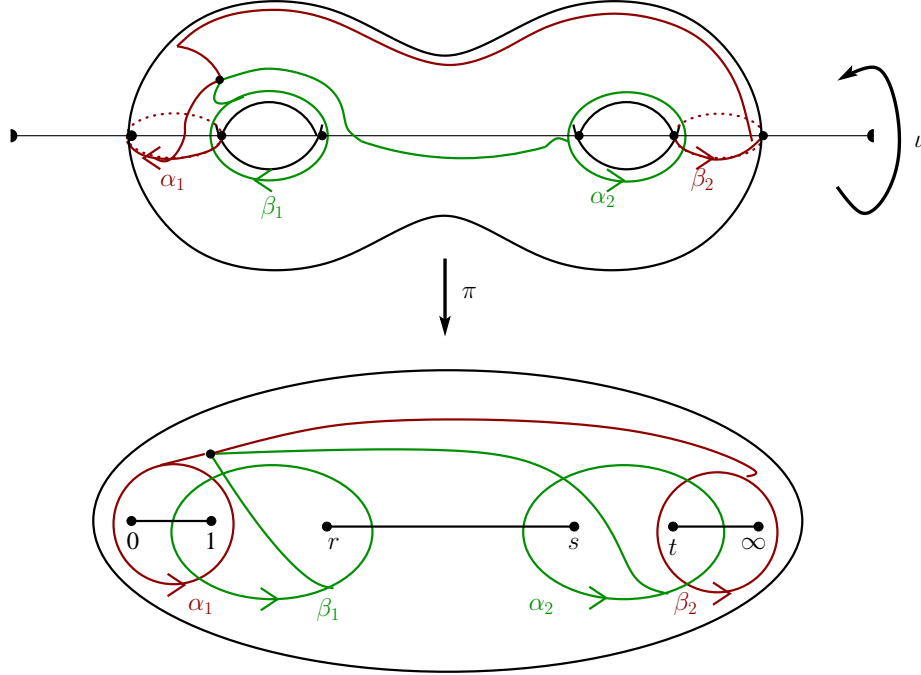


FIGURE 1. Elements of $\pi_1(\mathbb{P}^1 \setminus \underline{W}, x)$ that lift as the generators of $\pi_1(X, w)$.

After moving the base point to a Weierstrass point, $w = w_i$ say, the involution ι acts as an involutive automorphism of $\pi_1(X, w_i)$: it coincides with the outer automorphism given by γ_i -conjugacy. For instance, for $i = 1$, we get

$$\begin{cases} \alpha_1 \mapsto \alpha_1^{-1} \\ \beta_1 \mapsto \beta_1^{-1} \end{cases} \quad \begin{cases} \alpha_2 \mapsto \gamma\alpha_2^{-1}\gamma^{-1} \\ \beta_2 \mapsto \gamma\beta_2^{-1}\gamma^{-1} \end{cases} \quad \text{with } \gamma = \beta_1^{-1}\alpha_1^{-1}\beta_2\alpha_2.$$

Let us now prove (a) \Leftrightarrow (b). That irreducible representations are ι -invariant already appears in the last section of [24]. Let us recall the argument given there. There is a

natural surjective map

$$\Psi : \text{Hom}(\pi_1(X), \text{SL}_2) / \text{PGL}_2 \longrightarrow \text{Hom}(\pi_1(X), \text{SL}_2) // \text{PGL}_2 =: \chi$$

to the GIT quotient χ , usually called character variety, which is an affine variety. The singular locus is the image of reducible representations. There can be many different classes $[\rho]$ over each singular point. The smooth locus of χ however is the geometric quotient of irreducible representations, which are called stable points in this context. The above map Ψ is injective over this open subset. The involution ι acts on χ as a polynomial automorphism and we want to prove that the action is trivial. First note that the canonical fuchsian representation given by the uniformisation $\mathbb{H} \rightarrow X$ must be invariant by the hyperelliptic involution $\iota : X \rightarrow X$. The corresponding point in χ therefore is fixed by ι . On the other hand, the definition of χ only depends on the topology of X and, considering all possible complex structures on X , we now get a large set of fixed points $\chi_{\text{fuchsian}} \subset \chi$. Those fuchsian representations actually form an open subset of $\text{Hom}(\pi_1(X), \text{SL}_2\mathbb{R}) // \text{SL}_2\mathbb{R}$, and thus a Zariski dense subset of χ . It follows that the action of ι is trivial on the whole space χ . By injectivity of Ψ , any irreducible representation is ι -invariant.

In other words, if an irreducible representation ρ is defined by matrices $A_i, B_i \in \text{SL}_2$, $i = 1, 2$ with $[A_1, B_1] \cdot [A_2, B_2] = I_2$, then there exists $M \in \text{GL}_2$ satisfying:

$$(5) \quad \begin{cases} M^{-1}A_1M = A_1^{-1} \\ M^{-1}B_1M = B_1^{-1} \end{cases} \quad \begin{cases} M^{-1}A_2M = CA_2^{-1}C^{-1} \\ M^{-1}B_2M = CB_2^{-1}C^{-1} \end{cases} \quad \text{with} \quad C = B_1^{-1}A_1^{-1}B_2A_2.$$

Since the action of ι is involutive, M^2 commutes with ρ and is thus a scalar matrix. The matrix M has two opposite eigenvalues which can be normalized to ± 1 after replacing M by a scalar multiple. There are exactly two such normalizations, namely M and $-M$.

It remains to check what happens for reducible representations. In the strict reducible case (*i.e.* reducible but not diagonal), there is a unique common eigenvector for all matrices A_1, B_1, A_2, B_2 ; the representation ρ restricts to it as a representation $\pi_1(X) \rightarrow \mathbb{C}^*$ which must be ι -invariant. This (abelian) representation must therefore degenerate into $\{\pm 1\}$. It follows that any reducible ι -invariant representation is abelian. For abelian representations though, the action of ι is simply given by

$$A_i \mapsto A_i^{-1} \quad \text{and} \quad B_i \mapsto B_i^{-1} \quad \text{for} \quad i = 1, 2.$$

Hence all reducible ι -invariant representations are abelian and, up to conjugacy, we have:

- either A_1, B_1, A_2, B_2 are diagonal and one can choose $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,
- or A_1, B_1, A_2, B_2 are upper triangular with eigenvalues ± 1 (projectively unipotent) and $M = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ works.

Let us now prove (b) \Leftrightarrow (c). Given a representation $[\rho] \in \text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2) / \text{PGL}_2$, its image under Φ^{top} is ι -invariant, *i.e.* the action of ι coincides in this case with the conjugacy by $\rho(\gamma_1) \in \text{GL}_2$. Conversely, let $[\rho] \in \text{Hom}(\pi_1(X), \text{SL}_2) / \text{PGL}_2$ be ι -invariant, *i.e.* $\iota^*\rho = M^{-1} \cdot \rho \cdot M$ for some $M \in \text{GL}_2$ as in (5). From the cases discussed above, we know that M can be chosen with eigenvalues ± 1 . Then setting

$$\begin{cases} M_0 := A_1M \\ M_1 := M \\ M_r := B_1M \end{cases} \quad \begin{cases} M_s := B_2^{-1}A_1B_1M \\ M_t := A_1B_1MA_2B_2 \\ M_\infty := A_1B_1MA_2 \end{cases}$$

we get a preimage of $[\rho]$. The preimage depends only of the choice of M . Any other choice writes $M' := CM$ with C commuting with ρ . In the general case, *i.e.* when ρ is irreducible, we get two preimages given by M and $-M$. However, when ρ is diagonal, we get only one preimage, because the anti-diagonal matrices M and $-M$ are conjugated by a diagonal matrix (commuting with ρ). \square

Corollary 2.2. *The Galois involution of the double cover Φ^{top} is given by*

$$\left\{ \begin{array}{ccc} \text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2) / \text{PGL}_2 & \longrightarrow & \text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2) / \text{PGL}_2 \\ [\rho] & \mapsto & [-\rho] \end{array} \right\}.$$

So far, Theorem 2.1 provides an analytic description of the map Φ : although Φ^{top} is a polynomial branching cover, the Riemann-Hilbert correspondence is only analytic. In the next section, we will follow a more direct approach providing algebraic informations about Φ . However, note that we can already deduce the following:

Corollary 2.3. *An irreducible trace-free holomorphic connection (E, ∇) on X is invariant under the hyperelliptic involution: there exists a bundle isomorphism $h : E \rightarrow \iota^* E$ conjugating ∇ with $\iota^* \nabla$. We can moreover assume $h \circ \iota^* h = \text{id}_E$ and h is unique up to a sign.*

Remark 2.4. *Note that h acts as $-\text{id}$ on the determinant line bundle $\det(E) = \det(\iota^* E) \simeq \mathcal{O}$.*

Each Weierstrass point $w \in X$ is fixed by ι and the restriction of h to the fibre $E_w = \iota^* E_w$ is an automorphism with simple eigenvalues ± 1 .

2.1.1. Symmetry group. The 16-order group of 2-torsion characters $\text{Hom}(\pi_1(X), \{\pm 1\})$ acts on the space of representations $\text{Hom}(\pi_1(X), \text{SL}_2) / \text{PGL}_2$ by multiplication (changing signs of matrices A_i, B_i 's). This corresponds to the action of 2-torsion rank one connections on the moduli space $\mathfrak{Con}(X)$: the unique unitary connection on a 2-torsion line bundle is itself 2-torsion; if we twist a SL_2 -connection by this 2-torsion one, we get a new SL_2 -connection. Together with the involution of Corollary 2.2, we get a 32-order group acting on $\text{Hom}(\pi_1^{\text{orb}}(X/\iota), \text{GL}_2) / \text{PGL}_2$ also by changing signs of matrices M_i 's (only even change signs). The generators are described as follows

	(A_1, B_1, A_2, B_2)	$(M_0, M_1, M_r, M_s, M_t, M_\infty)$
$\mathcal{O}_X([w_0] - [w_1])$	$(-, +, +, +)$	$(-, -, +, +, +, +)$
$\mathcal{O}_X([w_1] - [w_r])$	$(+, -, +, +)$	$(+, -, -, +, +, +)$
$\mathcal{O}_X([w_s] - [w_t])$	$(+, +, -, +)$	$(+, +, +, -, -, +)$
$\mathcal{O}_X([w_t] - [w_\infty])$	$(+, +, +, -)$	$(+, +, +, +, -, -)$
\mathcal{O}_X	$(+, +, +, +)$	$(-, -, -, -, -, -)$

The quotient for this action identifies with one of the two connected components of $\text{Hom}(\pi_1(X), \text{PGL}_2) / \text{PGL}_2$, namely the component of those representations that lift to SL_2 . We have seen in Theorem 2.1 that the fixed point set of the Galois involution of Φ^{top} is given by diagonal representations. We can also compute the fixed point locus of $\mathcal{O}_X([w_0] - [w_1])$ for instance.

Proposition 2.5. *The fixed points of the action of $\mathcal{O}_X([w_0] - [w_1])$ (with its unitary connection) on the space of representations $\text{Hom}(\pi_1(X), \text{SL}_2) / \text{PGL}_2$ is parametrized by:*

$$A_1 = \pm I, \quad B_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \quad \text{and} \quad B_2 = \begin{pmatrix} b & 0 \\ 0 & b^{-1} \end{pmatrix}$$

with $(a, b) \in \mathbb{C}^* \times \mathbb{C}^*$.

2.2. A direct algebraic approach. Let (E, ∇) be a holomorphic trace-free rank 2 connection on X . As in Corollary 2.3, let h be a ∇ -invariant lift to the vector bundle E of the action of ι on X . Following [9] and [10], we can associate a parabolic logarithmic connection $(\underline{E}, \underline{\nabla}, \underline{p})$ on \mathbb{P}^1 with polar divisor \underline{W} and a natural choice of parabolic weights $\underline{\mu}$. Let us briefly recall this construction. The isomorphism h induces a non-trivial involutive automorphism on the rank 4 bundle $\pi_* E$ on \mathbb{P}^1 . The spectrum of such an automorphism is $\{-1, +1\}$ with respective multiplicities 2, which yields a splitting $\pi_* E = \underline{E} \oplus \underline{E}'$ with \underline{E} denoting the h -invariant subbundle.

In local coordinates, the automorphism h acts on $\pi_* E$ in the following way. If $U \subset X$ is a sufficiently small open set outside of the critical points, we have $\Gamma(\pi(U), \pi_* E) = \Gamma(U, E) \oplus \Gamma(\iota(U), E)$ and h permutes both direct summands. Locally at a Weierstrass point with local coordinate y , one can choose sections e_1 and e_2 generating E such that $h(e_1) = e_1$ and $h(e_2) = -e_2$ (recall that h has eigenvalues ± 1 in restriction to the Weierstrass fiber). On the corresponding open set of \mathbb{P}^1 , the bundle $\pi_* E$ is generated by $\langle e_1, e_2, ye_1, ye_2 \rangle$, and we see that $\langle e_1, ye_2 \rangle$ spans the h -invariant subspace. Since the connection ∇ on E is h -invariant, we can choose the sections e_1 and e_2 above to be horizontal for ∇ . Then considering the basis $\underline{e}_1 = e_1$ and $\underline{e}_2 = ye_2$ of \underline{E} , we get

$$\underline{\nabla} \underline{e}_1 = \nabla e_1 = 0 \quad \text{and} \quad \underline{\nabla} \underline{e}_2 = \nabla ye_2 = dy \otimes e_2 = \frac{dy}{y} \otimes \underline{e}_2 = \frac{1}{2} \frac{dx}{x} \otimes \underline{e}_2$$

so that $\underline{\nabla}$ is logarithmic with eigenvalues 0 and $\frac{1}{2}$. To each pole in \underline{W} , we associate the parabolic \underline{p}_i defined by the eigenspace with eigenvalue $\frac{1}{2}$, with the natural (in the sense of [10]) parabolic weight $\underline{\mu}_i = \frac{1}{2}$.

$$\begin{array}{ccccccc} (\tilde{E}, \tilde{\nabla}, \tilde{p}) & \xleftarrow{\text{elm}_W^-} & (E, \nabla, p) & \xleftarrow{h} & (E, \nabla) & \xrightarrow{-h} & (E, \nabla, p') & \xleftarrow{\text{elm}_W^-} & (\tilde{E}', \tilde{\nabla}', \tilde{p}') \\ \pi^* \uparrow & & & & \downarrow \pi_* & & & & \uparrow \pi^* \\ (\underline{E}, \underline{\nabla}, \underline{p}) & \xleftarrow{\text{eig}_{\frac{1}{2}}} & (\underline{E}, \underline{\nabla}) & \xrightarrow{\quad} & (\underline{E}, \underline{\nabla}) \oplus (\underline{E}', \underline{\nabla}') & \xleftarrow{\quad} & (\underline{E}', \underline{\nabla}') & \xleftarrow{\text{eig}_{\frac{1}{2}}} & (\underline{E}', \underline{\nabla}', \underline{p}') \\ & & & & \xleftarrow{\sqrt{d \log(W)} \otimes \text{elm}_W^+} & & & & \end{array}$$

Table 1: Hyperelliptic descent, lift and involution.

However, since we consider the rank 2 case, this general construction can also be viewed in the following way (summarized in Table 1): Denote by p the parabolic structure on E defined by the h -invariant directions over $W = \{w_0, w_1, w_r, w_s, w_t, w_\infty\}$ and associate the natural homogenous weight $\mu = 0$. In the coordinates above, the basis $(\underline{e}_1, \underline{e}_2)$ generates the vector bundle E after one negative elementary transformation in that direction. Now the hyperelliptic involution acts trivially on the parabolic logarithmic connection on X defined by

$$(\tilde{E}, \tilde{\nabla}, \tilde{p}, \tilde{\mu}) := \text{elm}_W^-(E, \nabla, p, \mu)$$

and we have

$$(\tilde{E}, \tilde{\nabla}, \tilde{p}, \tilde{\mu}) = \pi^*(\underline{E}, \underline{\nabla}, \underline{p}, \underline{\mu}).$$

2.2.1. Galois involution and symmetry group. With the notations above, let $(\underline{E}', \underline{\nabla}')$ be the connection on \mathbb{P}^1 we obtain for the other possibility of a lift of the hyperelliptic involution on $(E \rightarrow X, \nabla)$, namely for $h' = -h$. It is straightforward to check that the map from $(\underline{E}', \underline{\nabla}', \underline{p}')$ to $(\underline{E}, \underline{\nabla}, \underline{p})$ and vice-versa is obtained by the elementary transformations elm_W^+ over \mathbb{P}^1 , followed by the tensor product with a certain logarithmic rank 1 connection $\sqrt{d \log(\underline{W})}$ over \mathbb{P}^1 we now define:

There is a unique rank 1 logarithmic connection (L, ζ) on \mathbb{P}^1 having polar divisor \underline{W} and eigenvalues 1; note that $L = \mathcal{O}_{\mathbb{P}^1}(-6)$. We denote by $d \log(\underline{W})$ this connection and by $\sqrt{d \log(\underline{W})}$ its unique square root. In a similar way, define $\sqrt{d \log(\underline{D})}$ for any even order subdivisor $\underline{D} \subset \underline{W}$.

The Galois involution of our map $\Phi : \mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X)$ is therefore given by

$$\sqrt{d \log(\underline{W})} \otimes \text{elm}_W^+ : \mathfrak{Con}(X/\iota) \rightarrow \mathfrak{Con}(X/\iota).$$

There is a 16-order group of symmetries on $\mathfrak{Bun}(X)$ (resp. $\mathfrak{Con}(X)$) consisting of twists with 2-torsion line bundles (resp. rank 1 connections). It can be lifted as a 32-order group of symmetries on $\mathfrak{Bun}(X/\iota)$ (resp. $\mathfrak{Con}(X/\iota)$), namely those transformations $\sqrt{d \log(\underline{D})} \otimes \text{elm}_D^+$ with $\underline{D} \subset \underline{W}$ even. For instance, if $\underline{D} = w_i + w_j$, then its action on $\mathfrak{Con}(X/\iota)$ corresponds via Φ to the twist by the 2-torsion connection on $\mathcal{O}_X(w_i + w_j - K_X)$. In particular, it permutes the two parabolics (of \underline{p} and \underline{p}') over w_i and w_j .

3. FLAT VECTOR BUNDLES OVER X

In this section, we provide a description of the space of trace-free holomorphic connections on a given flat rank 2 vector bundle E over the genus 2 curve X . We first review the classical construction of the moduli space of semi-stable such bundles due to Narasimhan and Ramanan. We then present the *special* (in the sense of flat but not stable) bundles and explain how they arise in the Narasimhan-Ramanan moduli space. The 16-order group of 2-torsion points of $\text{Jac}(X)$ is naturally acting on $\mathfrak{Bun}(X)$ by tensor product, preserving each type of bundle. We compute this action for explicit coordinates in Section 3.6 along with explicit equations of the Kummer surface of strictly semi-stable bundles. Moreover, we describe the set of connections on each of these bundles and the quotient of the irreducible ones by the automorphism group. This is summarized in Table 2; columns list for each type of bundle the projective part $\mathbb{P}\text{Aut}(E) = \text{Aut}(E)/\mathbb{G}_m$ of the bundle automorphism group, the affine space of connections and lastly the moduli space of irreducible connections up to bundle automorphism. Here \mathbb{G}_m denotes the multiplicative group $(\mathbb{C}^\times, \cdot)$ and \mathbb{G}_a denotes the additive group $(\mathbb{C}, +)$.

Crucial for the understanding of the rest of the present paper is the case of Gunning bundles, where we explain the notion of two vector bundles being arbitrarily close in $\mathfrak{Bun}(X)$, which, as we will see, is responsible for the classical geometry of the Kummer surface of strictly semi-stable bundles in \mathcal{M}_{NR} .

3.1. Flatness criterion. Recall the well-known flatness criterion for vector bundles over curves [58, 1].

Theorem 3.1 (Weil). *A holomorphic vector bundle on a compact Riemann surface is flat, i.e. it admits a holomorphic connection, if and only if it is the direct sum of indecomposable bundles of degree 0.*

bundle type	E	$\mathbb{P}\text{Aut}(E)$	connections	moduli
stable	E	1	\mathbb{A}^3	\mathbb{A}^3
decomposable	$L_0 \oplus L_0^{-1}$	\mathbb{G}_m	\mathbb{A}^4	$\mathbb{C}^2 \times \mathbb{C}^*$
affine	$L_0 \rightarrow E \rightarrow L_0^{-1}$	1	\mathbb{A}^3	\emptyset
trivial+twists	E_0, E_τ	$\text{PGL}_2(\mathbb{C})$	\mathbb{A}^6	$\mathbb{C}_{(\nu_0, \nu_1, \nu_2)}^3 \setminus \{\nu_1^2 = 4\nu_0\nu_2\}$
unipotent+twists	$\tau \rightarrow E \rightarrow \tau$	\mathbb{G}_a	\mathbb{A}^4	$\mathbb{C}^2 \times \mathbb{C}^*$
Gunning	$\vartheta \rightarrow E_\vartheta \rightarrow \vartheta^{-1}$	$H^0(X, \Omega_X^1)$	\mathbb{A}^5	\mathbb{A}^3

Table 2: Bundle automorphisms and moduli spaces of irreducible connections.

In our case of rank 2 vector bundles E over a genus 2 curve X with trivial determinant bundle $\det(E) = \mathcal{O}_X$, Weil's criterion demands that either E is indecomposable, or it is the direct sum of degree 0 line bundles. We get the following list of flat bundles:

- stable bundles (forming a Zariski-open subset of the moduli space),
- decomposable bundles of the form $E = L \oplus L^{-1}$ where $L \in \text{Jac}(X)$ is a degree 0 line bundle,
- strictly semi-stable indecomposable bundles,
- Gunning bundles.

We recall that a *Gunning bundle* over X is an unstable indecomposable rank 2 vector bundle with trivial determinant bundle. There are precisely 16 such bundles: for each of the 16 line bundles $L \in \text{Pic}^1(X)$ such that $L^{\otimes 2} = \mathcal{O}(K_X)$ there is a unique indecomposable extension $0 \rightarrow L \rightarrow E \rightarrow L^{-1} \rightarrow 0$ of L^{-1} by L .

Given a flat bundle E , and a \mathfrak{sl}_2 -connection ∇ on E , any other \mathfrak{sl}_2 -connection writes

$$\nabla' = \nabla + \theta$$

where $\theta \in H^0(\text{Hom}_{\mathcal{O}_X}(\mathfrak{sl}(E) \otimes \Omega_X^1))$ is a Higgs field. Here, $\mathfrak{sl}(E)$ denotes the vector bundle whose sections are trace-free endomorphisms of E . On the other hand, by the Riemann-Roch Theorem and Serre Duality we have

$$(6) \quad h^0(\mathfrak{sl}(E) \otimes \Omega_X^1) = 3 \cdot \text{genus}(X) - 3 + h^0(\mathfrak{sl}(E))$$

($\mathfrak{sl}(E)$ is self-dual). Since there is no natural choice for the initial connection ∇ , the set of connections on E is an affine space. We will see in the following that for generic bundles we have $h^0(\mathfrak{sl}(E)) = 0$ and the *moduli space* of \mathfrak{sl}_2 -connections on E is \mathbb{A}^3 in this case. There are, however, flat bundles with non-trivial automorphisms for which the *moduli space* of \mathfrak{sl}_2 -connections will be a quotient of some \mathbb{A}^n by the automorphism group, yet the dimension of this quotient is always 3, as suggested by (6).

3.2. Semi-stable bundles and the Narasimhan-Ramanan theorem. Two semi-stable vector bundles of same rank and degree over a curve are called *S-equivalent*, if the graded bundles associated to Jordan-Hölder filtrations of these bundles are isomorphic. In our case, *i.e.* rank 2 bundles with trivial determinant bundle, we get that

- two stable bundles are *S-equivalent* if and only if they are isomorphic;
- two strictly semi-stable bundles are *S-equivalent* if and only if there is a line bundle $L \in \text{Jac}(X)$ such that each of the two bundles is an extension either of L^{-1} by L or of L by L^{-1} .

To a semi-stable bundle E , we associate (following [49]) the set

$$C_E = \{L \in \text{Pic}^1(X) \mid h^0(X, E \otimes L) > 0\}.$$

Equivalently, $L \in C_E$ if and only if there is a non-trivial (and thus injective) homomorphism $L^{-1} \rightarrow E$ of locally free sheaves. For stable bundles, the quotient E/L^{-1} then is necessarily locally free and hence defines an embedding of the total space of L^{-1} into the total space of E . The set C_E then parametrizes line subbundles of degree -1 .

Narasimhan and Ramanan proved that this set C_E is the support of a uniquely defined effective divisor D_E on $\text{Pic}^1(X)$ linearly equivalent to 2Θ , where

$$\Theta = \{[p] \mid p \in X\} \subset \text{Pic}^1(X)$$

is the locus of effective divisors of degree 1, naturally parametrized by the curve X itself. Moreover, for strictly semi-stable bundles, the divisor D_E only depends on the Jordan-Hölder filtration, *i.e.* on the S -equivalence class of E . We thus get a map

$$\text{NR} : \mathcal{M}_{\text{NR}} \rightarrow \mathbb{P}(\text{H}^0(\text{Pic}^1(X), \mathcal{O}(2\Theta)))$$

from the moduli space of S -equivalence classes to the linear system $|2\Theta|$ on $\text{Pic}^1(X)$.

Theorem 3.2 (Narasimhan-Ramanan). *The map NR defined above is an isomorphism. Let $\pi : \mathcal{E} \rightarrow T$ be a smooth family of semi-stable rank 2 vector bundles with trivial determinant over X . Then the map $\phi : T \rightarrow \mathcal{M}_{\text{NR}}$ associating to $t \in T$ the S -equivalence class of $E_t = \pi^{-1}(t)$ is a morphism.*

In particular, the moduli space of stable bundles naturally identifies with a Zariski open proper subset of $\mathcal{M}_{\text{NR}} \simeq \mathbb{P}^3$. A stable bundle has no non-trivial automorphism: we have $\text{Aut}(E) = \mathbb{C}^*$ acting by scalar multiplication in the fibres (see [28], thm 29). Therefore, *the moduli space of holomorphic connections $\nabla : E \rightarrow E \otimes \Omega_X^1$ on a given stable bundle E is an \mathbb{A}^3 -affine space.* Note that all holomorphic connections on a stable bundle are irreducible.

3.3. Semi-stable decomposable bundles. Let $E = L_0 \oplus L_0^{-1}$ with $L_0 \in \text{Jac}(X)$. Given $L \in \text{Pic}^1(X)$, non-trivial sections of $E \otimes L$ come from non-trivial sections of $L_0 \otimes L$ or $L_0^{-1} \otimes L$. We promptly deduce that

$$D_E = L_0 \cdot \Theta + L_0^{-1} \cdot \Theta$$

where $L_0 \cdot \Theta$ denotes the translation of Θ by L_0 for the group law on $\text{Pic}(X)$. A special case occurs for the 16 torsion points $L_0^2 = \mathcal{O}_X$ for which $L_0 = L_0^{-1}$ and hence $D_E = 2(L_0 \cdot \Theta)$ is not reduced.

The moduli space of semi-stable decomposable bundles naturally identifies with the *Kummer variety*

$$\text{Kum}(X) := \text{Jac}(X) / \iota,$$

the quotient of the Jacobian $\text{Jac}(X)$ by the involution $\iota : L_0 \mapsto \iota^* L_0 = L_0^{-1}$. The Narasimhan-Ramanan classifying map provides a canonical embedding

$$\text{NR} : \text{Kum}(X) := \text{Jac}(X) / \iota \hookrightarrow \mathcal{M}_{\text{NR}}$$

and the image is a quartic surface in $\mathcal{M}_{\text{NR}} \simeq \mathbb{P}^3$. The moduli space of stable bundles identifies with the complement of this surface. The 16 torsion points $L_0^2 = \mathcal{O}_X$ of the Jacobian are precisely the fixed points of the involution ι and yield 16 conic singularities on $\text{Kum}(X)$.

3.3.1. *The 2-dimensional family of decomposable bundles.* When $L_0^2 \neq \mathcal{O}_X$, the corresponding rank 2 bundle $E = L_0 \oplus L_0^{-1}$ lies on the smooth part of $\text{Kum}(X)$. Non-scalar automorphisms come from the independent action of \mathbb{G}_m on the two direct summands: we get a \mathbb{G}_m -action on $\mathbb{P}(E)$.

Given a connection on L_0 , we easily deduce a totally reducible connection ∇_0 on E (preserving both direct summands). Any other connection will differ from ∇_0 by a Higgs bundle: $\nabla = \nabla_0 + \theta$ where $\theta : E \rightarrow E \otimes \Omega_X^1$ is \mathcal{O}_X -linear and may be represented in the matrix way

$$\theta = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \begin{cases} \alpha : L_0 \rightarrow L_0 \otimes \Omega_X^1, \\ \beta : L_0^{-1} \rightarrow L_0 \otimes \Omega_X^1, \\ \gamma : L_0 \rightarrow L_0^{-1} \otimes \Omega_X^1. \end{cases}$$

Under our assumption that $L_0^2 \neq \mathcal{O}_X$, our space of connections is parametrized by $\mathbb{C}_\alpha^2 \times \mathbb{C}_\beta^1 \times \mathbb{C}_\gamma^1$. Since E has no degree 0 subbundle other than L_0 and L_0^{-1} , reducible connections on E are precisely those for which one of the two direct summands is invariant, *i.e.* $\beta = 0$ or $\gamma = 0$. The \mathbb{G}_m -action is trivial on α but not on the two other coefficients: the quotient $\mathbb{C}_\beta^1 \times \mathbb{C}_\gamma^1 / \mathbb{G}_m$ is \mathbb{C}^* after deleting reducible connections (for which $\beta = 0$ or $\gamma = 0$). *The moduli space of irreducible connections on E is thus given by $\mathbb{C}^2 \times \mathbb{C}^*$.*

The involution ι preserves those connections that are irreducible or totally reducible. *The moduli space of ι -invariant connections is $\mathbb{C}^2 \times \mathbb{C}$.*

3.3.2. *The trivial bundle and its 15 twists.* All 16 special decomposable bundles are equivalent to the trivial one after twisting by a convenient line bundle. Let us study the case $E = \mathcal{O}_X \oplus \mathcal{O}_X$ which admits the trivial connection $\nabla_0 = d$. Any other connection is obtained by adding a Higgs bundle of the matrix form

$$\theta = \begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \quad \text{with} \quad \alpha, \beta, \gamma \in H^0(X, \Omega_X^1)$$

(here, a trivialization of E is chosen). Our space of connections is parametrized by $\mathbb{C}_\alpha^2 \times \mathbb{C}_\beta^2 \times \mathbb{C}_\gamma^2$ but now the group acting is $\text{Aut}(\mathbb{P}E) = \text{PGL}_2$.

Since the trivial connection is PGL_2 -invariant, the data of a connection $\nabla = d + \theta$ is equivalent to the data of the Higgs field θ itself. Moreover, the determinant map

$$\det : H^0(\mathfrak{sl}(E) \otimes \Omega_X^1) \rightarrow H^0(\Omega_X^1 \otimes \Omega_X^1) ; \theta \mapsto -(\alpha \otimes \alpha + \beta \otimes \gamma)$$

is invariant under the PGL_2 -action. Through this map, we claim the following.

Proposition 3.3. *The moduli space of irreducible trace-free connections on the trivial bundle of rank 2 over X coincides with the open set in $H^0(X, \Omega_X^1 \otimes \Omega_X^1)$ of those quadratic differentials that are not the square $\omega \otimes \omega$ of a holomorphic 1-form ω .*

Proof. Note that in our usual coordinates on X , we have

$$H^0(X, \Omega_X^1) = \text{Vect}_{\mathbb{C}} \left(\frac{dx}{y}, x \frac{dx}{y} \right).$$

The eigendirections of θ define a curve C on the total space $X \times \mathbb{P}^1$ of the projectivized trivial bundle (for eigendirections to make sense, we have to compose θ by local isomorphisms $E \otimes \Omega_X^1 \rightarrow E$; the resulting curve C does not depend on this choice). In a concrete way, for each vector $v \in E$, we compute the determinant $v \wedge \theta(v)$. Under trivializing coordinates $(1 : z) \in \mathbb{P}^1$ we find that C is defined by

$$C : -\gamma + 2z\alpha + z^2\beta = 0.$$

It follows that C has bidegree $(2, 2)$ in $X \times \mathbb{P}^1$ (*i.e.* with respect to the variables (y, z)) and is invariant by the hyperelliptic involution ι . Hence it defines a bidegree $(1, 2)$ curve $\underline{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ (*i.e.* with respect to the variables (x, z)). It is easy to check that C is reducible if and only if ∇ is reducible. In the irreducible case, the curve C defines a $(2 : 1)$ -map $\mathbb{P}_z^1 \rightarrow \mathbb{P}_x^1$ whose Galois involution may be normalized to $z \mapsto -z$ under the PGL_2 -action. After this normalization, we get that $\alpha = 0$ and the involution ι lifts as $(x, y, z) \mapsto (x, -y, -z)$. In particular, $z = 0$ and $z = \infty$ are the two ι -invariant subbundles. This normalization is unique up to action of the dihedral group \mathbb{D}_∞ (preserving $z \in \{0, \infty\}$). Clearly, the determinant $-\beta \otimes \gamma$ is invariant and determines ∇ up to this action since, in genus 2, any quadratic form splits as a product $\det(\theta) = -\beta \otimes \gamma$. Finally, one can easily check that the following properties are equivalent:

- ∇ (or θ) is reducible,
- the $(1, 2)$ curve \underline{C} splits,
- the determinant $\det(\theta)$ viewed on \mathbb{P}_x^1 has a double zero,
- the determinant $\det(\theta)$ (viewed on X) is a square.

□

It may be of interest to pursue the discussion of the proof above in the reducible case. In this case, C is reducible and has a bidegree $(0, 1)$ -factor which is ∇ -invariant. We can normalize

$$\theta = \begin{pmatrix} \alpha & \beta \\ 0 & -\alpha \end{pmatrix}.$$

The gauge freedom is given by the group of upper-triangular matrices and we are led to the following cases

- (1) $\alpha \neq 0$ and β is not proportional to α (in particular $\neq 0$); the monodromy is *affine* but *non-abelian* and the curve \underline{C} splits as a union of irreducible bidegree $(0, 1)$ and $(1, 1)$ curves.
- (2) $\alpha \neq 0$ and β is proportional to α : we can normalize $\beta = 0$; the monodromy is *diagonal* and the curve \underline{C} splits as a union of two bidegree $(0, 1)$ curves and one $(1, 0)$ curve located at the vanishing point of α .
- (3) $\alpha = 0$ and $\beta \neq 0$; the monodromy is *unipotent* but non-trivial and the curve \underline{C} splits as a union of a bidegree $(0, 1)$ curve with multiplicity 2 and a bidegree $(1, 0)$ curve located at the vanishing point of β .
- (4) $\alpha = 0$ and $\beta = 0$ and we get the *trivial* connection (the curve \underline{C} has vanishing equation and is not defined).

The determinant map \det defined above takes values in the set of quadratic differentials over X . Those are of the form

$$\nu = \frac{\nu_0 + \nu_1 x + \nu_2 x^2}{x(x-1)(x-r)(x-s)(x-t)} dx \otimes dx.$$

It is a square, say $\det(\theta) = -\alpha \otimes \alpha$, if and only if $\nu_1^2 = 4\nu_0\nu_2$. In this case, α is uniquely defined up to a sign. It follows that a fiber $\det^{-1}(\nu)$ of the determinant map above is

- a unique irreducible connection (up to PGL_2 -isomorphism) if $\nu_1^2 \neq 4\nu_0\nu_2$;
- the union of two reducible connections of type (1) (upper and lower triangular once α is fixed) and a reducible connection of type (2) over a smooth point of the cone $\nu_1^2 = 4\nu_0\nu_2$;
- the union of the trivial connection (4) and a 1-parameter family of reducible connections of type (3) over the singular point $\nu_0 = \nu_1 = \nu_2 = 0$.

The moduli space of ι -invariant connections on the trivial bundle thus is not separated. Note that we obtain a double-cover of the moduli space of ι -invariant but non trivial connections on the trivial bundle by considering the family of connections of the form

$$d + \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix},$$

where one of the coefficients $\beta_0, \beta_1, \gamma_0, \gamma_1 \in \mathbb{C}$ is normalized to 1 (with the obvious transition maps). Here $\beta = \beta_0 \frac{dx}{y} + \beta_1 x \frac{dx}{y}$ and $\gamma = \gamma_0 \frac{dx}{y} + \gamma_1 x \frac{dx}{y}$.

3.4. Semi-stable indecomposable bundles. In this case, the bundle is a non-trivial extension $0 \rightarrow L_0 \rightarrow E \rightarrow L_0^{-1} \rightarrow 0$ for some $L_0 \in \text{Jac}(X)$. It is S -equivalent in the sense of Narasimhan-Ramanan to the corresponding trivial extension. For fixed L_0 , the moduli space of such extensions is isomorphic to $\mathbb{P}H^1(X, L_0^2)$ which, by Serre duality, identifies with $\mathbb{P}H^0(X, L_0^{-2} \otimes \Omega_X^1)$. Again, the discussion splits into two cases.

3.4.1. The 1-dimensional family of unipotent bundles and its 15 twists. When $L_0^2 = \mathcal{O}_X$, the moduli space of non-trivial extensions $0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0$ is parametrized by $\mathbb{P}H^1(X, \mathcal{O}_X) \simeq \mathbb{P}H^0(X, \Omega_X^1) \simeq \mathbb{P}^1$; we call *unipotent bundles* such bundles E . Following [44], the automorphism group of E is $\text{Aut}(E) = \mathbb{G}_m \ltimes \mathbb{G}_a$. The action of \mathbb{G}_a is faithful in restriction to each fiber E_w , unipotent and fixing the unique subbundle $\mathcal{O}_X \subset E$; the action of \mathbb{G}_m is scalar as usual.

For a convenient open covering (U_i) of X , the bundle E is defined by a matrix cocycle of the form

$$M_{ij} = \begin{pmatrix} 1 & b_{ij} \\ 0 & 1 \end{pmatrix}$$

where $(b_{ij}) \in H^1(X, \mathcal{O}_X)$ is a non trivial scalar cocycle. Moreover, from the short exact sequence

$$0 \rightarrow H^0(X, \Omega_X^1) \rightarrow H^1(X, \mathbb{C}) \rightarrow H^1(X, \mathcal{O}_X) \rightarrow 0,$$

(b_{ij}) may be lifted to $H^1(X, \mathbb{C})$, so that E is flat: the local connections d_X over U_i glue together to form a global connection (non-canonical) ∇_0 with unipotent monodromy. Conversely, if a connection (E, ∇) has unipotent monodromy, defined by say

$$A_1 = \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, B_1 = \begin{pmatrix} 1 & b_1 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}, B_2 = \begin{pmatrix} 1 & b_2 \\ 0 & 1 \end{pmatrix}$$

(with respect to the standard basis (4)), then E is either the trivial bundle, or a unipotent bundle; in fact, we are in the former case if, and only if, (a_1, b_1, a_2, b_2) is the period data of a holomorphic 1-form on X .

Proposition 3.4. *Let ∇_0 be a unipotent connection on E like above. Then the general connection on E can be described as*

$$\nabla = \nabla_0 + \lambda_1 \theta_1 + \lambda_2 \theta_2 + \lambda_3 \theta_3 + \lambda_4 \theta_4$$

with $(\lambda_i) \in \mathbb{C}^4$ so that the \mathbb{G}_a -action is given by

$$\left(c, \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \right) \longrightarrow \begin{pmatrix} \lambda_1 \\ \lambda_2 - c\lambda_1 \\ \lambda_3 + 2c\lambda_2 - c^2\lambda_1 \\ \lambda_4 \end{pmatrix}$$

Moreover, reducible (resp. unipotent) connections are given by $\lambda_1 = 0$ (resp. $\lambda_1 = \lambda_2 = 0$). The moduli space of irreducible connections on E identifies with $\mathbb{C}^* \times \mathbb{C}^2$.

Proof. A general trace-free connection on E is defined by a collection

$$d + \theta_i \quad \text{where} \quad \theta_i = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & -\alpha_i \end{pmatrix}$$

are matrices of holomorphic 1-forms on U_i satisfying the compatibility condition

$$\theta_j = M_{ij}^{-1} \theta_i M_{ij} + M_{ij}^{-1} dM_{ij}$$

on $U_i \cap U_j$ or, equivalently,

$$(7) \quad \begin{cases} \alpha_i - \alpha_j &= b_{ij} \gamma_i \\ \beta_i - \beta_j &= -2b_{ij} \alpha_i + b_{ij}^2 \gamma_i \\ \gamma_i - \gamma_j &= 0 \end{cases}$$

When $\alpha_i = \gamma_i = 0$, we precisely obtain all those connections with unipotent monodromy on E ; the second equation (7) then tells us that (β_i) defines a global holomorphic 1-form $\beta \in H^0(X, \Omega_X^1)$.

When $\gamma_i = 0$, we get all reducible connections on E . The first equation (7) tells us that (α_i) defines a global holomorphic 1-form $\alpha \in H^0(X, \Omega_X^1)$. To solve the second equation (7), we need that the image under Serre duality

$$\begin{array}{ccc} H^1(X, \mathcal{O}_X) \times H^0(X, \Omega_X^1) & \rightarrow & H^1(X, \Omega_X^1) \xrightarrow{\sim} \mathbb{C} \\ (b_{ij}, \alpha) & \mapsto & (b_{ij} \alpha) \end{array}$$

is the zero cocycle. In other words, α must belong to the orthogonal $(b_{ij})^\perp$ (with respect to Serre duality). In this case, we can solve (β_i) , and the solution is unique up to addition by a global holomorphic 1-form β .

Irreducible connections occur for $\gamma \neq 0$ (note that the third equation (7) states that (γ_i) is a global 1-form). Then, the first equation (7) imposes that $\gamma \in (b_{ij})^\perp$ (the orthogonal for Serre duality). Therefore, the collection (α_i) solving the cocycle $(b_{ij} \gamma)$ is unique up to the addition of a global holomorphic 1-form $\alpha \in H^0(X, \Omega_X^1)$. Finally, to solve the second equation in (β_i) , we have to insure that the cocycle

$$(-2b_{ij} \alpha_i + b_{ij}^2 \gamma) \in H^1(X, \Omega_X^1)$$

is zero, which can be achieved by conveniently using the freedom α when solving the first equation. Precisely, if we add some global 1-form α to the collection (α_i) , then we translate the previous cocycle by $(-2b_{ij} \alpha)$. For a convenient choice of α (or (α_i)), the cocycle becomes trivial. Note that we still have the freedom to add any 1-form α belonging to the orthogonal $(b_{ij})^\perp$. At the end, we can find a solution (β_i) which is unique up to addition by a global holomorphic 1-form $\beta \in H^0(X, \Omega_X^1)$.

Given an irreducible connection as above, defined by (α_i) , (β_i) and $\gamma \neq 0$, and given a global holomorphic 1-form $\beta \notin (b_{ij})^\perp$, it follows from above case-by-case discussion that any connection ∇ on E takes the form

$$d_X + \lambda_1 \begin{pmatrix} \alpha_i & \beta_i \\ \gamma & -\alpha_i \end{pmatrix} + \lambda_2 \begin{pmatrix} \gamma & -2\alpha_i \\ 0 & -\gamma \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 & \gamma \\ 0 & 0 \end{pmatrix} + \lambda_4 \begin{pmatrix} 0 & \beta \\ 0 & 0 \end{pmatrix}$$

over charts U_i , for convenient scalars λ_i . Unipotent bundle automorphisms are given in these charts by a constant matrix $\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix}$, with $c \in \mathbb{C}$ not depending on U_i , and it is straightforward to check that the action on λ_i is the one of the statement. \square

3.4.2. Affine bundles. When $L_0^2 \neq \mathcal{O}_X$, then $\mathbb{P}H^0(X, L_0^{-2} \otimes \Omega_X^1)$ reduces to a single point: there is only one non-trivial extension of L_0^{-1} by L_0 up to isomorphism. Following [44], the automorphism group of such a bundle E is $\text{Aut}(E) = \mathbb{C}^*$. In particular, we have $\mathbb{P}\text{Aut}(E) = \{1\}$ and *the space of holomorphic connections on E up to automorphisms is an affine \mathbb{A}^3 space with homogeneous part $H^0(\mathfrak{sl}(E) \otimes \Omega_X^1) \simeq \mathbb{C}^3$.*

A curious phenomenon occurs for affine bundles E : *all connections on E are reducible, none of them is totally reducible.* Indeed, L_0 is the unique subbundle of degree 0, but is not invariant by the hyperelliptic involution. Therefore, the vector bundle E itself is not ι -invariant. This implies that the monodromy of a connection ∇ on E can be neither irreducible, nor totally reducible. Note that this phenomenon does not occur for higher genus (see [32], Prop. (3.3), p.70). Note further that even if affine bundles do not allow hyperelliptic descent, we can see them in smooth charts of the moduli space of flat bundles using Tyurin's approach (see Section 4.1).

3.5. Unstable and indecomposable: the 6 + 10 Gunning bundles. There are 16 theta characteristics, i.e. square-roots of $\Omega_X^1 = \mathcal{O}_X(K_X)$. They split into

- 6 odd theta characteristics $\mathcal{O}_X([w_i])$, $i = 0, 1, r, s, t, \infty$;
- 10 even theta characteristics $\mathcal{O}_X([w_i] + [w_j] - [w_\infty])$, $i \neq j \neq \infty$.

Given a theta characteristic ϑ , there is a unique non-trivial extension $0 \rightarrow \vartheta \rightarrow E_\vartheta \rightarrow \vartheta^{-1} \rightarrow 0$ up to isomorphism, which is called the *Gunning bundle* E_ϑ associated to ϑ . We talk about *even* or *odd* Gunning bundle depending on the nature of ϑ . We have $\text{Aut}(E_\vartheta) \simeq \mathbb{G}_m \ltimes H^0(X, \Omega_X^1)$ (see [44]); the group $H^0(X, \Omega_X^1)$ is acting by unipotent bundle automorphisms on fibers $E|_w$, fixing the subbundle ϑ .

A connection ∇ on E necessarily satisfies Griffiths transversality with respect to the flag $0 \subset \vartheta \subset E_\vartheta$ and defines an "oper" (see [5]). Following [29], the data of ∇ up to automorphism of E is equivalent to the data of a projective structure on X . Moreover, any two projective structures differ on X by a quadratic differential: once a projective structure has been chosen, the moduli space identifies with $H^0(X, \mathcal{O}_X(2K_X))$. However, there is no natural choice of "origin", i.e. there is no canonical projective structure on X from an algebraic point of view (see [40]). *The moduli space of (irreducible) connections on E_ϑ is therefore an \mathbb{A}^3 -affine space.*

Recall that the Narasimhan-Ramanan classifying map is defined only for semi-stable bundles and thus not for Gunning bundles. This has the following geometric reason: We say two rank 2 vector bundles E and E' with trivial determinant over X are *arbitrarily close* if there are smooth families of vector bundles $(E_t)_{t \in \mathbb{A}^1}$ and $(E'_t)_{t \in \mathbb{A}^1}$ over X such that $E_t = E'_t$ for each $t \neq 0$ and $E_0 = E$, $E'_0 = E'$. By the Narasimhan-Ramanan-theorem, two arbitrarily close semi-stable vector bundles are S -equivalent. It turns out that the Gunning bundle E_ϑ is arbitrarily close to any semi-stable extension $0 \rightarrow \vartheta^{-1} \rightarrow E' \rightarrow \vartheta \rightarrow 0$ (see Proposition 3.5). These are precisely the semi-stable bundles whose corresponding divisor $D_{E'} \sim 2\Theta$ (see Section 3.2) passes through the point ϑ on $\text{Pic}^1(X)$. They define a 2-plane in \mathcal{M}_{NR} which we will call *Gunning plane* and denote it by Π_ϑ .

The intersection of Π_ϑ with the Kummer surface is easily described as

$$\Pi_\vartheta \cap \text{Kum}(X) = \{L_0 \oplus L_0^{-1} \mid L_0 \in \vartheta^{-1} \cdot \Theta\}.$$

In fact, the 16 Gunning planes Π_ϑ are well-known; each of them is tangent to the Kummer surface along a conic passing through 6 of the 16 nodes. The above description gives a natural parametrization of the hyperelliptic cover of this marked conic by the curve X itself (via the Θ divisor). Precisely, for each Π_ϑ , the 6 corresponding nodes

are those parametrized by the 2-torsion points $\vartheta^{-1} \otimes \mathcal{O}([w_i])$ where w_i runs over the six Weierstrass points. Conversely, through each node pass 6 of the 16 planes. This so-called $(16, 6)$ configuration is classical (see [34, 26]) as well as the interpretation in terms of the moduli space of vector bundles (see [49, 14]). However, the interpretation of Π_ϑ in terms of semi-stable bundles arbitrarily close to Gunning bundles seems to be new so far.

Proposition 3.5. *Given two extensions*

$$0 \rightarrow L \rightarrow E_0 \rightarrow L' \rightarrow 0 \quad \text{and} \quad 0 \rightarrow L' \rightarrow E'_0 \rightarrow L \rightarrow 0$$

of the same (but permuted) line bundles, there are two deformations E_t and E'_t of these bundles (parametrized by \mathbb{A}^1) such that $E_t \simeq E'_t$ for $t \neq 0$.

Proof. The vector bundles E_0 and E'_0 are respectively defined by a cocycle of the form

$$(8) \quad \begin{pmatrix} a_{ij} & b_{ij} \\ 0 & d_{ij} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{ij} & 0 \\ c_{ij} & d_{ij} \end{pmatrix}$$

for a convenient open covering (U_i) of X . Here, (a_{ij}) and (d_{ij}) are cocycles respectively defining L and L' . We claim that this can be achieved with only two Zariski open sets $X = U_1 \cup U_2$ so that we can neglect the cocycle condition. Before proving this claim, let us show how to conclude the proof. Consider the deformations E_t and E'_t respectively defined by

$$\begin{pmatrix} a_{ij} & b_{ij} \\ tc_{ij} & d_{ij} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} a_{ij} & tb_{ij} \\ c_{ij} & d_{ij} \end{pmatrix}.$$

They define the same vector bundle up to isomorphism for $t \neq 0$ since these cocycles are conjugated by the automorphism of $L \oplus L'$ defined in the matrix way by $\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}$. On the other hand, we clearly have $E_t \rightarrow E_0$ and $E'_t \rightarrow E'_0$ when $t \rightarrow 0$. For a general open covering, these matrices fail to satisfy the cocycle condition $A_{ij}A_{jk}A_{ki} = I$; this is why we need to work with only two open sets.

Although it might be standard, let us prove the claim. Up to tensoring by a very ample line bundle $\tilde{L} = \mathcal{O}_X(\tilde{D})$, we may assume that L, L', E_0 and E'_0 are all generated by global holomorphic sections. Choose one such section s_1 for L ; it is then easy to construct another section s_2 such that the corresponding (effective) divisors D_1 and D_2 have disjoint support. Indeed, given any non-zero section s_2 , for any common zero with s_1 one can find some section s non-vanishing at that point: one can then perturb $s_2 := s_2 + \epsilon \cdot s$. This means that L may be trivialized on each open set $U_i = X \setminus \text{supp}(D_i)$, $i = 1, 2$, and therefore defined with respect to this covering by a single cocycle a_{12} . In a similar way, we can construct sections σ_1 and σ_2 of E_0 such that the two sections $s_i \wedge \sigma_i$ of $\det(E_0)$ have disjoint zeroes. In other words, possibly by deleting more points in the open sets U_i , the vector bundle E_0 can simultaneously be trivialized on each of these open sets, and is therefore defined by a cocycle of the above form. To deal simultaneously with L' and E'_0 , the easiest way is to consider the zero set of sections $s_i \wedge \sigma_i \wedge s'_i \wedge \sigma'_i$ of $\det(E_0 \oplus E'_0)$. Finally, the same manipulation can be done with sections \tilde{s}_i of the ample bundle \tilde{L} : considering the zeros of sections $s_i \wedge \sigma_i \wedge s'_i \wedge \sigma'_i \wedge \tilde{s}_i$ of $\det(E_0 \oplus E'_0 \oplus \tilde{L})$ we can assume that the sections σ_i, s_i, s'_i and \tilde{s}_i have no common zeroes for $i = 1, 2$. Tensoring with $\tilde{L}^{\otimes -1}$ we have constructed bases (s_i, σ_i) (resp. (σ'_i, s'_i)) of $E|_{U_i}$ (resp. $E'|_{U_i}$) with $i = 1, 2$ such that the corresponding cocycles of E and E' respectively are of the form (8). \square

In particular, two semi-stable rank 2 bundles are arbitrarily close if and only if they are S -equivalent.

3.6. Computation of a system of coordinates. In this section, we construct coordinates on the Narasimhan-Ramanan moduli space allowing us to express explicitly the Kummer surface of strictly semi-stable bundles as well as the involutions of the moduli space given by tensor products with 2-torsion line bundles.

For all computations, the curve X is the smooth compactification of the affine complex curve defined by

$$X : y^2 = x(x-1)(x-r)(x-s)(x-t)$$

where $0, 1, r, s, t \in \mathbb{C}$ are pair-wise distinct; we denote by ∞ the point at infinity.

Let us first calculate a basis of $H^0(\text{Pic}^1(X), \mathcal{O}_X(2\Theta))$ in order to introduce explicit projective coordinates on the three-dimensional projective space

$$\mathbb{P}_{\text{NR}}^3 := \mathbf{PH}^0(\text{Pic}^1(X), \mathcal{O}_X(2\Theta)).$$

Since $\text{Pic}^1(X)$ is birationally equivalent to the symmetric product $X^{(2)}$, rational functions on $\text{Pic}^1(X)$ can be conveniently expressed as symmetric rational functions on $X \times X$.

$$\begin{array}{ccc} X \times X & \longrightarrow & X^{(2)} \xrightarrow{\phi^{(2)}} \text{Pic}^2(X) \xrightarrow{\sim} \text{Pic}^1(X) \\ & & \{P, Q\} \longmapsto [P] + [Q] \end{array}$$

The pull-back of the divisor $\Theta \subset \text{Pic}^1(X)$ (resp. $\Theta + [\infty] \subset \text{Pic}^2(X)$) to $X \times X$ is $\overline{\Delta} + \infty_1 + \infty_2$, where

- $\overline{\Delta}$ is the anti-diagonal $\{(P, Q) \in X \times X \mid Q = \iota(P)\}$,
- ∞_1 is the divisor $\{\infty\} \times X$ and
- ∞_2 the divisor $X \times \{\infty\}$.

The pull-back to $X \times X$ of 2Θ , viewed as a divisor on $\text{Pic}^1(X)$ is then (see Figure 2):

$$2\overline{\Delta} + 2\infty_1 + 2\infty_2.$$

Lemma 3.6. *Let $(P_1, P_2) = ((x_1, y_1), (x_2, y_2))$ be coordinates of $X \times X$. Then*

$$H^0(X \times X, \mathcal{O}_X^{\text{sym}}(2\overline{\Delta} + 2\infty_1 + 2\infty_2)) = \text{Vect}_{\mathbb{C}}(1, \text{Sum}, \text{Prod}, \text{Diag})$$

with

$$\begin{aligned} 1 : (P_1, P_2) &\mapsto 1 \\ \text{Sum} : (P_1, P_2) &\mapsto x_1 + x_2 \\ (9) \quad \text{Prod} : (P_1, P_2) &\mapsto x_1 x_2, \\ \text{Diag} : (P_1, P_2) &\mapsto \left(\frac{y_2 - y_1}{x_2 - x_1} \right)^2 - (x_1 + x_2)^3 + (1 + \sigma_1)(x_1 + x_2)^2 + \\ &\quad + x_1 x_2 (x_1 + x_2) - (\sigma_1 + \sigma_2)(x_1 + x_2) \end{aligned}$$

where σ_1, σ_2 and σ_3 are the following constants: $\sigma_1 = r + s + t, \sigma_2 = rs + st + tr, \sigma_3 = rst$.

Proof. We have $h^0(X \times X, \mathcal{O}_X^{\text{sym}}(2\overline{\Delta} + 2\infty_1 + 2\infty_2)) = h^0(\text{Pic}^1(X), \mathcal{O}(2\Theta)) = 4$ (see [49] or [47]). The function *Diag* can be rewritten as

$$(10) \quad \text{Diag} = \frac{1}{(x_1 - x_2)^2} \cdot \left[-2y_1 y_2 - 2(1 + \sigma_1)x_1^2 x_2^2 - (\sigma_2 + \sigma_3)(x_1^2 + x_2^2) + (x_1 + x_2) \cdot (x_1^2 x_2^2 + (\sigma_1 + \sigma_2)x_1 x_2 + \sigma_3) \right]$$

The expression of *Diag* in (9) shows that it has no poles off the anti-diagonal and the infinity (and in particular no poles on the diagonal). From the expression (10) follows

easily that $Diag$ has polar divisor $2\overline{\Delta} + 2\infty_1 + 2\infty_2$. Indeed, if u_1 is the local parameter for X_1 near ∞_1 defined by $x_1 = \frac{1}{u_1^2}$, then the principal part of the generating functions is given by

$$1, \quad Sum = \frac{1}{u_1^2} + x_2, \quad Prod = \frac{x_2}{u_1^2} \quad \text{and} \quad Diag \sim \frac{x_2^2}{u_1^2} - \frac{y_2}{u_1} + \dots$$

As a section of $H^0(\text{Pic}^1(X), \mathcal{O}_X(2\Theta))$, the function 1 vanishes twice along Θ while the other ones do not vanish identically on Θ . \square

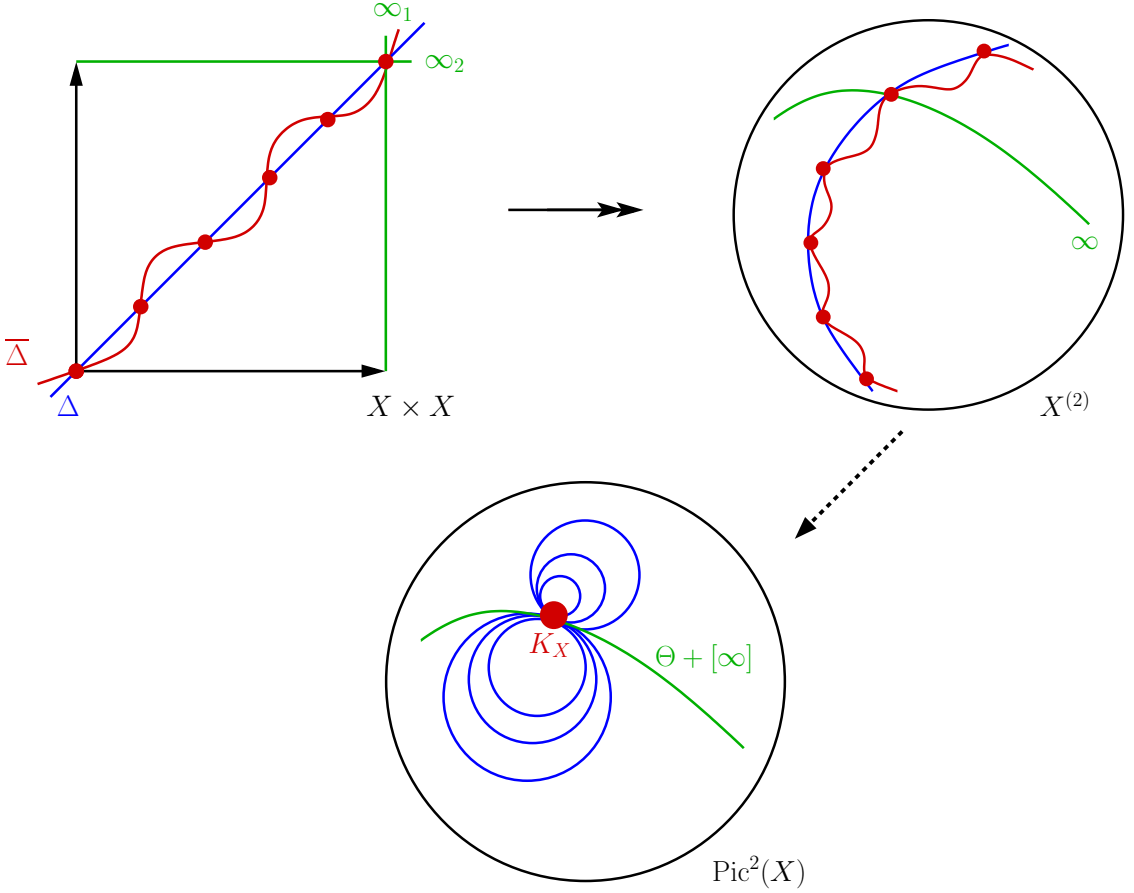


FIGURE 2. X^2 as a rational cover of $\text{Jac}(X)$.

In the sequel, denote by $(v_0 : v_1 : v_2 : v_3)$ the projective coordinate on \mathbb{P}_{NR}^3 representing the function

$$v_0 + v_1 \cdot Sum + v_2 \cdot Prod + v_3 \cdot Diag.$$

In order to compute the strictly semi-stable locus, namely the Kummer surface embedded in \mathcal{M}_{NR} , it is enough to consider the image in \mathbb{P}_{NR}^3 of decomposable semi-stable bundles. Let $L = \mathcal{O}_X([P_1] + [P_2] - [\infty]) \in \text{Pic}^1(X)$ be a line bundle such that $L^2 \neq \mathcal{O}_X(K_X)$ and denote by \tilde{L} the associated degree 0 bundle $\tilde{L} = \mathcal{O}_X([P_1] + [P_2] - 2[\infty])$. Let us now calculate the explicit coordinates of the corresponding Narasimhan-Ramanan

divisor $\tilde{L} \cdot \Theta + \tilde{L}^{-1} \cdot \Theta$ on $\text{Pic}^1(X)$, which is linearly equivalent to the divisor 2Θ (see Section 3.3). The first component $\tilde{L} \cdot \Theta$ is parametrized by

$$X \rightarrow \text{Pic}^1(X) ; Q \mapsto [\underline{P}_1] + [\underline{P}_2] + [Q] - 2[\infty].$$

Setting $[\underline{P}_1] + [\underline{P}_2] + [Q] - 2[\infty] \sim [P_1] + [P_2] - [\infty]$, we get that $[\underline{P}_1] + [\underline{P}_2] + [Q]$ belongs to the linear system $[P_1] + [P_2] + [\infty]$. This latter one is generated by the two functions 1 and $f(P) := \frac{y+y_1}{x-x_1} - \frac{y+y_2}{x-x_2}$ on the curve. Therefore, $[P_1] + [P_2] - [\infty] \in \tilde{L} \cdot \Theta$ (the support of) if, and only if, $f(\underline{P}_1) = f(\underline{P}_2)$; this gives the following equation for $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$:

$$\frac{\underline{y}_1 + y_1}{\underline{x}_1 - x_1} - \frac{\underline{y}_1 + y_2}{\underline{x}_1 - x_2} = \frac{\underline{y}_2 + y_1}{\underline{x}_2 - x_1} - \frac{\underline{y}_2 + y_2}{\underline{x}_2 - x_2}.$$

The equation for the other component $\tilde{L}^{-1} \cdot \Theta$ is deduced by changing signs $\underline{y}_i \rightarrow -\underline{y}_i$ for $i = 1, 2$. Taking into account the two equations, we get an equation for $\tilde{L} \cdot \Theta + \tilde{L}^{-1} \cdot \Theta$:

$$\left(\frac{\underline{y}_1 + y_1}{\underline{x}_1 - x_1} - \frac{\underline{y}_1 + y_2}{\underline{x}_1 - x_2} - \frac{\underline{y}_2 + y_1}{\underline{x}_2 - x_1} + \frac{\underline{y}_2 + y_2}{\underline{x}_2 - x_2} \right) \left(\frac{-\underline{y}_1 + y_1}{\underline{x}_1 - x_1} - \frac{-\underline{y}_1 + y_2}{\underline{x}_1 - x_2} - \frac{-\underline{y}_2 + y_1}{\underline{x}_2 - x_1} + \frac{-\underline{y}_2 + y_2}{\underline{x}_2 - x_2} \right) = 0$$

which, after reduction, writes

$$(11) \quad -\text{Diag}(\underline{P}_1, \underline{P}_2) \cdot 1 + \text{Prod}(\underline{P}_1, \underline{P}_2) \cdot \text{Sum} - \text{Sum}(\underline{P}_1, \underline{P}_2) \cdot \text{Prod} + 1 \cdot \text{Diag} = 0$$

Remark 3.7. *The symmetric form of this equation is due to the fact that for any vector bundle $E \in \mathcal{M}_{\text{NR}}$ and any line bundle $L \in \text{Pic}^1(X)$ such that $h^0(X, E \otimes L) > 0$, the divisor D_E associated to E and the divisor $\tilde{L} \cdot \Theta + \tilde{L}^{-1} \cdot \Theta$ associated to $\tilde{L} \oplus \tilde{L}^{-1}$ intersect precisely in L and $\iota(L)$ on $\text{Pic}^1(X)$.*

Hence, according to equation (11), the Kummer embedding

$$\begin{aligned} \text{Jac}(X) &\rightarrow \text{Kum}(X) \subset \mathbb{P}_{\text{NR}}^3 \\ \mathcal{O}_X([\underline{P}_1] + [\underline{P}_2] - 2[\infty]) &\mapsto (v_0 : v_1 : v_2 : v_3) \end{aligned}$$

is explicitly given by

$$(12) \quad (v_0 : v_1 : v_2 : v_3) = (-\text{Diag}(\underline{P}_1, \underline{P}_2) : \text{Prod}(\underline{P}_1, \underline{P}_2) : -\text{Sum}(\underline{P}_1, \underline{P}_2) : 1)$$

One can now eliminate parameters \underline{P}_1 and \underline{P}_2 from (12) as follows: express $\underline{y}_1 \underline{y}_2$ in terms of functions $\underline{x}_1 + \underline{x}_2$ and $\underline{x}_1 \underline{x}_2$ and variable v_0/v_3 , so that the square can be replaced by

$$\left(\underline{y}_1 \underline{y}_2 \right)^2 = \prod_{w=0,1,r,s,t} (w^2 - (\underline{x}_1 + \underline{x}_2)w + \underline{x}_1 \underline{x}_2);$$

then replace $\underline{x}_1 \underline{x}_2$ and $\underline{x}_1 + \underline{x}_2$ by v_1/v_3 and $-v_2/v_3$ respectively. We get

$$\begin{aligned} \text{Kum}(X) : \\ 0 = & (v_0 v_2 - v_1^2)^2 \cdot 1 \\ & -2 [(\sigma_1 + \sigma_2)v_1 + (\sigma_2 + \sigma_3)v_2](v_0 v_2 - v_1^2) \\ & + 2(v_0 + \sigma_1 v_1)(v_0 + v_1)v_1 + 2(\sigma_2 v_1 + \sigma_3 v_2)(v_1 + v_2)v_1 \cdot v_3 \\ & -2\sigma_3(v_0 v_2 - v_1^2) + [(\sigma_1 + \sigma_2)^2 v_1 + (\sigma_2 + \sigma_3)^2 v_2](v_1 + v_2) \\ & -(\sigma_1 + \sigma_3)^2 v_1 v_2 + 4[(\sigma_2 + \sigma_3)v_0 - \sigma_3 v_2]v_1 \cdot v_3^2 \\ & -2\sigma_3 [(\sigma_1 + \sigma_2)v_1 - (\sigma_2 + \sigma_3)v_2] \cdot v_3^3 \\ & + \sigma_3^2 \cdot v_3^4. \end{aligned}$$

Here, we see that $v_3 = 0$ is a (Gunning-) plane tangent to $\text{Kum}(X)$ along a conic. In the formula above as in the following, we denote:

$$\sigma_1 = r + s + t, \quad \sigma_2 = rs + st + tr, \quad \sigma_3 = rst.$$

Following formula (12), we can compute the locus of the trivial bundle E_0 and its 15 twists $E_\tau := E_0 \otimes \mathcal{O}_X(\tau)$, where $\tau = [w_i] - [w_j]$ with $i \neq j$.

E_τ	$(v_0 : v_1 : v_2 : v_3)$
E_0	$(1 : 0 : 0 : 0)$
$E_{[w_0]-[w_\infty]}$	$(0 : 0 : 1 : 0)$
$E_{[w_1]-[w_\infty]}$	$(1 : -1 : 1 : 0)$
$E_{[w_r]-[w_\infty]}$	$(r^2 : -r : 1 : 0)$
$E_{[w_s]-[w_\infty]}$	$(s^2 : -s : 1 : 0)$
$E_{[w_t]-[w_\infty]}$	$(t^2 : -t : 1 : 0)$
$E_{[w_0]-[w_1]}$	$(rs + st + rt : 0 : -1 : 1)$
$E_{[w_0]-[w_r]}$	$(r(st + s + t) : 0 : -r : 1)$
$E_{[w_0]-[w_s]}$	$(s(rt + r + t) : 0 : -s : 1)$
$E_{[w_0]-[w_t]}$	$(t(rs + r + s) : 0 : -t : 1)$
$E_{[w_1]-[w_r]}$	$((1 + r)st : r : -1 - r : 1)$
$E_{[w_1]-[w_s]}$	$((1 + s)rt : s : -1 - s : 1)$
$E_{[w_1]-[w_t]}$	$((1 + t)rs : t : -1 - t : 1)$
$E_{[w_r]-[w_s]}$	$((r + s)t : rs : -r - s : 1)$
$E_{[w_r]-[w_t]}$	$((r + t)s : rt : -r - t : 1)$
$E_{[w_s]-[w_t]}$	$((s + t)r : st : -s - t : 1)$

The Gunning planes Π_ϑ are the planes passing through 6 of these 16 singular points. Precisely, the odd Gunning plane with $\vartheta = [w_i]$ is passing through all E_τ with $\tau = [w_i] - [w_j]$ (including the trivial bundle E_0 for $i = j$); for an even Gunning plane with $\vartheta = [w_i] + [w_j] - [w_k] \sim [w_l] + [w_m] - [w_n]$, where $\{i, j, k, l, m, n\} = \{0, 1, r, s, t, \infty\}$, we get

$$\left. \begin{matrix} E_{[w_i]-[w_j]}, E_{[w_j]-[w_k]}, E_{[w_i]-[w_k]} \\ E_{[w_l]-[w_m]}, E_{[w_m]-[w_n]}, E_{[w_l]-[w_n]} \end{matrix} \right\} \in \Pi_{[w_i]+[w_j]-[w_k]} = \Pi_{[w_l]+[w_m]-[w_n]}.$$

In particular, we can derive explicit equations, for instance:

$\Pi_{[w_0]}$	$v_1 = 0$
$\Pi_{[w_1]}$	$v_1 + v_2 + v_3 = 0$
$\Pi_{[w_\infty]}$	$v_3 = 0$
$\Pi_{[w_0]+[w_1]-[w_\infty]}$	$v_0 + v_1 = (rs + st + rt)v_3$

We can also compute the 16-order linear group given by twisting the general bundle E by a 2-torsion line bundle $\mathcal{O}_X(\tau)$, $\tau = [w_i] - [w_j]$, by looking at the induced permutation on Kummer's singular points. For instance, we get

$$\begin{aligned} (v_0 : v_1 : v_2 : v_3) &\xrightarrow{\otimes E_{[w_0]-[w_\infty]}} ((\sigma_2 + \sigma_3)v_1 + \sigma_3v_2 : \sigma_3v_3 : v_0 - (\sigma_2 + \sigma_3)v_3 : v_1) \\ (v_0 : v_1 : v_2 : v_3) &\xrightarrow{\otimes E_{[w_1]-[w_\infty]}} (v_0 : v_1 : v_2 : v_3) \cdot \begin{pmatrix} 1 & \sigma_1 + \sigma_3 & \sigma_2 & 0 \\ -1 & -1 & 0 & \sigma_2 \\ 1 & 0 & -1 & -(\sigma_1 + \sigma_3) \\ 0 & 1 & 1 & 1 \end{pmatrix}^T \end{aligned}$$

One can find in [34, 26] classical equations for Kummer surfaces which are nicer than the above one, but they require coordinate changes that are non-rational in (r, s, t) . For instance, we can choose $E_0, E_{[w_0]-[w_1]}, E_{[w_1]-[w_\infty]}$ and $E_{[w_0]-[w_\infty]}$ as a projective frame so that the Gunning bundles $\Pi_{[w_0]}, \Pi_{[w_1]}, \Pi_{[w_\infty]}$ and $\Pi_{[w_0]+[w_1]-[w_\infty]}$ become coordinate hyperplanes. The Kummer equation therefore becomes quadratic in each coordinate.

However, to reach the nice form given in §54 (page 83) of [34], we must choose square roots $\alpha^2 = rst$ and $\beta^2 = (r-1)(s-1)(t-1)$. Then, setting

$$(u_0 : u_1 : u_2 : u_3) = ((v_0 + v_1 - \sigma_2 v_3) : \beta v_1 : \alpha(v_1 + v_2 + v_3) : \alpha \beta v_3),$$

we get the new equation

Kum(X) :

$$\begin{aligned} 0 = & (u_0^2 u_3^2 + u_1^2 u_2^2) + \beta^2 (u_0^2 u_2^2 + u_1^2 u_3^2) + \alpha^2 (u_0^2 u_1^2 + u_2^2 u_3^2) \\ & - 2\beta (u_0 u_2 - u_1 u_3) (u_0 u_3 + u_1 u_2) - 2\alpha (u_0 u_3 - u_1 u_2) (u_0 u_1 - u_2 u_3) \\ & - 2\alpha \beta (u_0 u_1 + u_2 u_3) (u_0 u_2 + u_1 u_3) - 2(\sigma_1 + \sigma_2 - 2\sigma_3 - 2) u_0 u_1 u_2 u_3. \end{aligned}$$

In these coordinates, the translations computed above simply become:

$$\begin{aligned} (u_0 : u_1 : u_2 : u_3) & \xrightarrow{\otimes E_{[w_0] - [w_\infty]}} (u_2 : u_3 : u_0 : u_1) \\ (u_0 : u_1 : u_2 : u_3) & \xrightarrow{\otimes E_{[w_1] - [w_\infty]}} (u_1 : -u_0 : -u_3 : u_2). \end{aligned}$$

Another classical presentation of the Kummer surface consists in normalizing the action of the finite translation group to be generated by double-transpositions of variables and double-changes of signs. Then the equation of the Kummer surface takes the very nice form (see §53 page 80-81 of [34])

$$(13) \quad \begin{aligned} & (t_0^4 + t_1^4 + t_2^4 + t_3^4) + 2D(t_0 t_1 t_2 t_3) \\ & + A(t_0^2 t_3^2 + t_1^2 t_2^2) + B(t_1^2 t_3^2 + t_0^2 t_2^2) + C(t_2^2 t_3^2 + t_0^2 t_1^2) = 0 \end{aligned}$$

with coefficients A, B, C, D satisfying the following relation

$$4 - A^2 - B^2 - C^2 + ABC + D^2 = 0.$$

Note that any coordinate change commuting with the (already normalized) actions of $E_{[w_0] - [w_\infty]}$ and $E_{[w_1] - [w_\infty]}$ in the coordinates $(u_0 : u_1 : u_2 : u_3)$ takes the form

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ c & d & a & b \\ d & -c & -b & a \end{pmatrix} \cdot \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ u_3 \end{pmatrix}.$$

If, moreover, we want to normalize the action of all the translation group to the one given in the table below for example,

τ	$(t_0 : t_1 : t_2 : t_3) \otimes E_\tau$
0	$(t_0 : t_1 : t_2 : t_3)$
$[w_0] - [w_\infty]$	$(t_2 : t_3 : t_0 : t_1)$
$[w_1] - [w_\infty]$	$(t_1 : -t_0 : -t_3 : t_2)$
$[w_r] - [w_\infty]$	$(t_0 : -t_1 : -t_2 : t_3)$
$[w_s] - [w_\infty]$	$(t_1 : t_0 : -t_3 : -t_2)$
$[w_t] - [w_\infty]$	$(t_2 : t_3 : -t_0 : -t_1)$

then the variables a, b, c, d have to satisfy (up to a common factor):

$$\begin{aligned} a &= rst(r-s)\beta + t\gamma\delta - rt(r-1)\delta - st\beta\gamma \\ b &= -st(s-1)\gamma + rt\beta\delta \\ c &= t(r-s)\alpha\beta - t(r-1)\alpha\delta \\ d &= -t(r-1)(s-1)(r-s)\alpha + t(s-1)\alpha\gamma \end{aligned}$$

where $\alpha, \beta, \gamma, \delta$ satisfy

$$\alpha^2 = rst, \beta^2 = (r-1)(s-1)(t-1), \gamma^2 = r(r-1)(r-s-(r-t)) \text{ and } \delta^2 = s(s-1)(s-r)(s-t).$$

The coefficients of the resulting Kummer equation (13) are

$$(14) \quad \begin{aligned} A &= -2 \frac{s(t-1)+(t-s)}{t(s-1)} & B &= -2 \frac{r+(r-t)}{t} \\ C &= 2 \frac{(r-1)+(r-s)}{s-1} & D &= -4 \frac{r(s-t)+(r-s)}{t(s-1)}. \end{aligned}$$

The five t -polynomials occuring in the Kummer equation (13) are fundamental invariants for the action of the translation group and define a natural map $\mathbb{P}_{\text{NR}}^3 \rightarrow \mathbb{P}^4$ whose image is a quartic hypersurface (see [18], Proposition 10.2.7).

Corollary 3.8. *The quartic in \mathbb{P}^4 defined by the natural map $\mathbb{P}_{\text{NR}}^3 \rightarrow \mathbb{P}^4$ is a coarse moduli space of S -equivalence classes of semi-stable \mathbb{P}^1 -bundles over X .*

Remark 3.9. *Recall that a \mathbb{P}^1 -bundle S over X is called semi-stable if $\#(s, s) \geq 0$ for every section $s : X \rightarrow S$. If E is a rank 2 vector bundle over X such that $\mathbb{P}E = S$, then the (semi-)stability of S is equivalent to the semi-stability of E [4].*

Proof. Let T be a smooth parameter space and $\mathcal{S} \rightarrow X \times T$ a family of \mathbb{P}^1 -bundles over X . Denote by π_T the projection $X \times T \rightarrow T$. The \mathbb{P}^1 -bundle \mathcal{S} lifts to a rank 2 bundle $\mathcal{E} \rightarrow X \times T$ such that $\det(\mathcal{E}) = \pi_T^* \mathcal{O}_X$ and $\mathbb{P}\mathcal{E} = \mathcal{S}$. This vector bundle is unique up to tensor product with $\pi_T^*(L)$ where L is a 2-torsion line bundle on X . According to Theorem 3.2, the classification map $T \rightarrow \mathcal{M}_{\text{NR}}$ then is a morphism as is its composition with the natural map $\mathbb{P}_{\text{NR}}^3 \rightarrow \mathbb{P}^4$. The resulting morphism $T \rightarrow \mathbb{P}^4$ no longer depends on the choice of \mathcal{E} . \square

4. ANTICANONICAL SUBBUNDLES

We will now enrich our point of view of hyperelliptic decent by its relations to the classical approaches of Tyurin [56] (see also [36]) and Bertram [7], as well as more recent works of Bolognesi [14, 15]. By our main construction (see Section 2), we see $\mathfrak{Bun}(X/\iota)$ as the moduli space of hyperelliptic parabolic bundles (E, \mathbf{p}) together with the forgetful map

$$\mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X); (E, \mathbf{p}) \mapsto E.$$

The Bertram-Bolognesi point of view (see Section 4.2) arises from the moduli space of hyperelliptic flags (E, L) with $E \supset L \simeq \mathcal{O}(-K_X)$: Bertram considered in [7] the projective space of non-trivial extensions

$$0 \longrightarrow \mathcal{O}(-K_X) \longrightarrow E \longrightarrow \mathcal{O}(K_X) \longrightarrow 0$$

on which the hyperelliptic involution acts naturally. Bertram's moduli space is the invariant hyperplane, *i.e.* the set of hyperelliptic extensions.

Tyurin however considers rank 2 vector bundles with trivial determinant over X that can be obtained from $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$ by positive elementary transformations on a parabolic structure carried by a divisor in $|2K_X|$. Again we obtain a moduli space, which is a rational two-cover of an open set of $\mathfrak{Bun}(X)$. It will turn out later (see Section 5.6), that the moduli spaces for each of these points of view are all birationally equivalent.

Let E be a flat vector bundle with trivial determinant bundle on X . Given an irreducible connection ∇ on E , Corollary 2.3 provides a lift $h : E \rightarrow \iota^*E$ of the hyperelliptic involution $\iota : X \rightarrow X$ whose action on the Weierstrass fibers is non-trivial, with two distinct eigenvalues ± 1 . Consider the set of line subbundles $\mathcal{O}(-K_X) \hookrightarrow E$ and how h acts on it. In Section 4.1, we will prove that a generic $E \in \mathfrak{Bun}(X)$ carries a 1-parameter family of such subbundles, only two of them being h -invariant:

- $L^+ \subset E$ on which h acts as id_{L^+} ,
- $L^- \subset E$ on which h acts as $-\text{id}_{L^-}$.

In the generic case, the two parabolic structures \mathbf{p} and \mathbf{p}' discussed in Sections 2.2 and 2.2.1 are therefore respectively defined by the fibres over the Weierstrass points of the line subbundles L^+ and L^- of E . We also investigate the non generic case. The results are summarized in Table 3.

bundle type	degenerate invariant Tyurin subbundles	parabolic structures \mathbf{p}^\pm (up to autom.) determined by L^\pm
stable off Gunning planes	\emptyset	2 out of 2
generic on $\Pi_{[w_i]}$	$L^+ = \mathcal{O}_X(-[w_i])$	1 out of 2
stable on $\Pi_{[w_i]} \cap \Pi_{[w_j]}$	$L^+ = \mathcal{O}_X(-[w_i]), L^- = \mathcal{O}_X(-[w_j])$	0 out of 2
generic decomposable	\emptyset	1 out of 1
$L_0 \oplus L_0$ with $L^2 = \mathcal{O}_X$	$L^+ = L_0, L^- = L_0$	1 out of 1
generic unipotent	$L^+ = \mathcal{O}_X$	2 out of 2
special unipotent	$L^+ = \mathcal{O}_X, L^- = \mathcal{O}(-[w])$	1 out of 2
twists of unipotent	$L^+ = \mathcal{O}_X([w_i] - [w_j])$	1 out of 2
affine	\emptyset	0 out of 0
even Gunning bundle	$L^+ = L^- = \mathcal{O}_X(\vartheta)$	2 out of 2
odd Gunning bundle	$L^+ = \mathcal{O}_X(\vartheta)$	2 out of 2

Table 3: Invariant Tyurin subbundles for the different types of bundles. By definition non-degenerate subbundles are isomorphic to $\mathcal{O}_X(-K_X)$.

4.1. Tyurin subbundles. Let (E, ∇) be an irreducible trace-free connection over X , and let $h : E \rightarrow \iota^*E$ be the lift of the hyperelliptic involution $\iota : X \rightarrow X$ given by Corollary 2.3. Recall that h acts non-trivially with two distinct eigenvalues on each Weierstrass fiber $E|_w$. The involution ι acts linearly on $\mathcal{O}(-K_X)$ and therefore h acts on $H^0(\text{Hom}(\mathcal{O}(-K_X), E))$. Since it is involutive, this action induces a splitting

$$H^0(\text{Hom}(\mathcal{O}(-K_X), E)) = H^+ \oplus H^-$$

into eigenspaces (relative to ± 1 eigenvalues). We call *Tyurin subbundle* of E the line subbundles L obtained by saturation of the inclusion of locally trivial sheaves $\mathcal{O}(-K_X) \hookrightarrow E$

defined by any non zero element $\varphi \in H^0(\text{Hom}(\mathcal{O}(-K_X), E))$. In the following, we prefer to consider φ as a holomorphic map from the total space of $\mathcal{O}(-K_X)$ to the total space of E . From this point of view, if x_1, \dots, x_n are the points of X such that $\varphi|_{\mathcal{O}(-K_X)_{x_i}}$ is identically zero, then the corresponding Tyurin subbundle L satisfies $L \simeq \mathcal{O}_X([x_1] + \dots [x_n] - K_X)$. In other words, if φ is injective (as a map between total spaces of vector bundles), then $L \simeq \mathcal{O}_X(-K_X)$. We say that a Tyurin subbundle $L \subset E$ is *degenerate* if $L \not\simeq \mathcal{O}_X(-K_X)$.

Proposition 4.1. *Let E and h be as above. The vector space $H^0(\text{Hom}_{\mathcal{O}_X}(\mathcal{O}(-K_X), E))$ is 2-dimensional except in the following cases*

- E is either unipotent, or an odd Gunning bundle, and then the dimension is 3,
- E is the trivial bundle, and then the dimension is 4.

If E is not an even Gunning bundle, the images of these morphisms span the vector bundle E at a generic point. The two eigenspaces H^+ and H^- then have positive dimension; they correspond to morphisms into two distinct h -invariant subbundles, L^+ and L^- . There are no other h -invariant Tyurin subbundles.

Remark 4.2. *As we shall see in Section 4.1.6, in the case of even Gunning bundles, the eigenspaces H^+ and H^- still have positive dimension, but the associated h -invariant subbundles L^+ and L^- are equal.*

Proof. First we have $\text{Hom}(\mathcal{O}(-K_X), E) \simeq E \otimes \mathcal{O}_X(K_X)$ and by the Riemann-Roch formula $h^0(E \otimes \mathcal{O}_X(K_X)) - h^0(E) = 2$. Here, we use Serre duality and the fact that E is selfdual (because $\text{rank}(E) = 2$ and $\det(E) = \mathcal{O}_X$). We promptly deduce that $h^0(E \otimes \mathcal{O}_X(K_X)) \geq 2$ and > 2 if and only if E has non-zero sections or, equivalently, if it contains a subbundle L of the form $L = \mathcal{O}_X, \mathcal{O}_X([p])$ or $\deg(L) > 1$. Because of flatness (see Section 3), the only possibilities are actually $L = \mathcal{O}_X$ or $\mathcal{O}_X([w])$ for some Weierstrass point $w \in X$.

When the image of a 2-dimensional subspace of $H^0(\text{Hom}(\mathcal{O}_X(-K_X), E))$ is degenerate, *i.e.* contained in a proper subbundle $L \subset E$, then $h^0(L \otimes \mathcal{O}_X(K_X)) = 2$ which implies $L = \mathcal{O}_X$ or $L = \vartheta$, a theta characteristic. Yet in the cases when L is trivial or an odd theta characteristic, we have $h^0(\text{Hom}(\mathcal{O}_X(-K_X), E)) > 2 = h^0(\text{Hom}(\mathcal{O}_X(-K_X), L))$ and thus not all morphisms take values into L : we get enough freedom to span E at a generic point.

Now, given two morphisms $\varphi_i : \mathcal{O}_X(-K_X) \rightarrow E$ for $i = 1, 2$, taking values into two different subbundles $L_i \subset E$, $L_1 \neq L_2$, we get a morphism $\varphi_1 \oplus \varphi_2 : \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X) \rightarrow E$ whose image spans the vector bundle E at all fibers but those corresponding to the (effective) zero divisor of $\varphi_1 \wedge \varphi_2 : \mathcal{O}_X(-2K_X) \rightarrow \mathcal{O}_X$. Such a divisor takes the form $[P_1] + [\iota(P_1)] + [P_2] + [\iota(P_2)]$ for some $P_1, P_2 \in X$. We thus get an isomorphism between the 2-dimensional vector space $\text{Vect}_{\mathbb{C}}(\varphi_1, \varphi_2) \subset H^0(\text{Hom}(\mathcal{O}_X(-K_X), E))$ and the fiber of E over each point of $X \setminus \{P_1, \iota(P_1), P_2, \iota(P_2)\}$. In particular, over a Weierstrass point $w \neq P_1, P_2$, we have $E|_w \simeq \text{Vect}_{\mathbb{C}}(\varphi_1, \varphi_2)$ and since the action h on $H^0(\text{Hom}(\mathcal{O}_X(-K_X), E))$ is non-trivial, neither H^+ nor H^- is reduced to $\{0\}$. Moreover, φ_1 and φ_2 cannot belong to a common eigenspace of the action of h on $H^0(\text{Hom}(\mathcal{O}_X(-K_X), E))$. In other words, any two morphisms belonging to the same eigenspace H^{\pm} take image in the same subbundle, say L^{\pm} .

Let now L be a Tyurin subbundle distinct from L^+ and L^- : L is generated by $\varphi = \varphi_1 + \varphi_2$ for some $\varphi_1 \in H^+$ and $\varphi_2 \in H^-$. Again, there is a Weierstrass point w

where $\varphi_1 \wedge \varphi_2$ does not vanish: the action of h is homothetic on the φ_i with opposite eigenvalues and cannot fix the direction $\mathbb{C} \cdot \varphi(w)$. Thus L is not h -invariant. \square

Note that if the line subbundles L^\pm are non-degenerate, their fibres over the Weierstrass points define the parabolic structures \mathbf{p}^\pm . As we shall see, any flat vector bundle E has degenerate Tyurin subbundles; some of them can be h -invariant, even in the stable case.

In the following paragraphs, we will study the Tyurin subbundles for each type of bundle, following the list of Section 3. The reader might want to skip the non stable cases at first, and refer to them later, when needed.

4.1.1. Stable bundles. When E is stable, any holomorphic connection on E is irreducible. Since the only bundle automorphisms of E are homothecies, the same bundle isomorphism $h : E \rightarrow \iota^* E$ works for all connections and it therefore only depends on the bundle (up to a sign). The two h -invariant Tyurin bundles L^+ and L^- depend (up to permutation) only on E .

Consider two elements $\varphi^+, \varphi^- \in H^0(\text{Hom}(\mathcal{O}(-K_X), E))$ generating L^+ and L^- (at a generic point) and consider the divisor $\text{div}(\varphi^+ \wedge \varphi^-) = [P] + [\iota(P)] + [Q] + [\iota(Q)]$. This divisor $D_E^T \in |2K_X|$ is an invariant of the bundle, we call it the *Tyurin divisor*. Let $D_E \in |2\Theta|$ be the divisor on $\text{Pic}^1(X)$ defined by Narasimhan-Ramanan (see Section 3.2).

Proposition 4.3. *Let E be stable. Then the divisor D_E^T is the intersection between the divisor D_E and the natural embedding $X \rightarrow \Theta; P \mapsto [P]$ on $\text{Pic}^1(X)$:*

$$D_E^T = D_E \cdot \Theta.$$

For each point P of the support of D_E^T , there is exactly one subbundle $L_P \equiv \mathcal{O}_X(-[P])$ of E . These are precisely the degenerate Tyurin subbundles. Such a degenerate Tyurin subbundle L_P is h -invariant if, and only if, $P = w$ is a Weierstrass point. This happens precisely when E lies on the odd Gunning plane $\Pi_{[w]}$.

Proof. First note that $D_E^T = \text{div}(\varphi_1 \wedge \varphi_2)$ for any basis (φ_1, φ_2) of the vector space $H^0(\text{Hom}(\mathcal{O}(-K_X), E))$. A point $P \in X$ belongs to the support of D_E^T if and only if $\iota(P)$ does. This is equivalent to the fact that φ^+ and φ^- are colinear at $\iota(P)$. Equivalently, there is a morphism $\varphi_P \in H^0(\text{Hom}(\mathcal{O}(-K_X), E))$ which vanishes at $\iota(P)$ (and can be completed to a basis with φ^+ or φ^-). By stability of the vector bundle E , the morphism φ_P cannot vanish elsewhere. Denote by L_P the line subbundle corresponding to φ_P . Finally, we have $P \in D_E^T$ if and only if there is a line subbundle L_P of E such that $L_P \simeq \mathcal{O}([\iota(P)] - K_X) = \mathcal{O}(-[P])$. On the other hand, P belongs to the support of $D_E \cdot \Theta$ if and only if there is a line subbundle $L_P \simeq \mathcal{O}(-[P])$ of E . Since these divisors are generically reduced, we can conclude by continuity that $D_E^T = D_E \cdot \Theta$.

Now suppose E has two line subbundles L_P . A linear combination of the two corresponding homomorphisms in $H^0(\text{Hom}(\mathcal{O}(-K_X), E))$ then would have a double zero at P , which is impossible by stability of E . So for each point P in the support of D_E^T , we get a unique subbundle $L_P \simeq \mathcal{O}_X(-[P])$ and there are no other degenerate Tyurin subbundles.

Finally, note that the finite set of (at most 4) degenerate Tyurin subbundles must be h -invariant. Thus such a bundle L_P is invariant if, and only if, P is ι -invariant. \square

Corollary 4.4. *When E is stable and outside of odd Gunning planes $\Pi_{[w_i]}$, there are exactly two h -invariant subbundles $L^+, L^- \simeq \mathcal{O}(-K_X)$ in E that are invariant under the*

hyperelliptic involution h . The two parabolic structures \mathbf{p} and \mathbf{p}' defined in Sections 2.2 and 2.2.1 then are precisely the fibres over the Weierstrass points of two line subbundles L^\pm .

Another important consequence of the proposition above is the Tyurin parametrization of the moduli space of stable bundles (see section 4.3) which relies on the following

Corollary 4.5. *When E is stable and the Tyurin divisor $D_E^T = [P] + [\iota(P)] + [Q] + [\iota(Q)]$ is reduced (4 distinct points), then the natural map*

$$\varphi^+ \oplus \varphi^- : \mathcal{O}(-K_X) \oplus \mathcal{O}(-K_X) \rightarrow E$$

is a positive elementary transformation for the parabolic structure defined over $D_E^T = [P] + [\iota(P)] + [Q] + [\iota(Q)]$ by the fibres of the line subbundles L_P , $L_{\iota(P)}$, L_Q and $L_{\iota(Q)}$ over the corresponding points.

Remark 4.6. *When E belongs to an odd Gunning plane $\Pi_{[w]}$, then one of the two h -invariant Tyurin subbundles is degenerate, say $L^- = \mathcal{O}_X(-[w])$, and fails to determine the parabolic structure \mathbf{p}^- over the Weierstrass point w . When $E \in \Pi_{[w_i]} \cap \Pi_{[w_j]}$, then the two h -invariant Tyurin subbundles are degenerate and neither \mathbf{p}^+ , nor \mathbf{p}^- are determined by these bundles.*

4.1.2. Generic decomposable bundles. Let $E = L_0 \oplus L_0^{-1}$, where $L_0 = \mathcal{O}([P] + [Q] - K_X)$ is not 2-torsion: $L_0^2 \neq \mathcal{O}_X$. There is (up to scalar multiple) a unique morphism $\varphi : \mathcal{O}(-K_X) \rightarrow L_0$ (resp. $\varphi' : \mathcal{O}(-K_X) \rightarrow L_0^{-1}$) vanishing at $[P] + [Q]$ (resp. $[\iota(P)] + [\iota(Q)]$). They generate all Tyurin subbundles and they are the only degenerate ones. Clearly, neither L_0 nor L_0^{-1} is invariant. The projective part \mathbb{G}_m of the automorphism group $\text{Aut}(E)$ fixes both L_0 and L_0^{-1} and acts transitively on the remaining part of the family. Any involution h interchanges L_0 and L_0^{-1} while it fixes two generic members L^+ and L^- of the family. The parabolic structures are defined by the fibres of these two bundles over the Weierstrass points. Another choice of lift $h' = g \circ h \circ g^{-1}$, $g \in \text{Aut}(E)$, just translates the two subbundles L^\pm by g . Finally, *up to automorphism, there is a unique invariant Tyurin bundle, and thus a unique parabolic structure.*

4.1.3. The trivial bundle and its 15 twists. When E is the trivial bundle, the space of morphisms $H^0(\text{Hom}(\mathcal{O}(-K_X), E))$ is 4-dimensional and generated by 2-dimensional subspaces $H^0(\text{Hom}(\mathcal{O}(-K_X), \mathcal{O}_X))$ for two distinct embeddings $\mathcal{O}_X \hookrightarrow E$. We get 1-parameter family of degenerate Tyurin subbundles formed by all embeddings $\mathcal{O}_X \hookrightarrow E$. Given any irreducible connection, the corresponding lift h fixes only two (degenerate) subbundles (see Section 3.3.2). The two parabolic structures are defined by the line bundles associated to these two embeddings $\mathcal{O}_X \hookrightarrow E$. Therefore, *up to automorphism, there is exactly one parabolic structure on the trivial vector bundle.*

When $E = L_0 \oplus L_0$ with $L_0 = \mathcal{O}([w_i] + [w_j] - K_X)$ and $i \neq j$, then the vector space $H^0(\text{Hom}(\mathcal{O}(-K_X), E))$ is 2-dimensional and all Tyurin subbundles are degenerate in this case: they form the 1-parameter family of subbundles $L \hookrightarrow E$ with $L \simeq L_0$. Still $\text{Aut}(E) = \text{GL}_2(\mathbb{C})$ acts transitively on them. Recall that the two parabolic structures \mathbf{p}^\pm on E can be deduced from the case of trivial bundles just by permuting the role of the two parabolics over w_i and w_j with respect to \mathbf{p}^+ and \mathbf{p}^- (see Section 2.2.1). Each parabolic structure is thus distributed on two embeddings $L \hookrightarrow E$ with $L \simeq L_0$. Since $\text{Aut}(E)$ acts 2-transitively on the family of such line subbundles, *there is a unique parabolic structure up to automorphisms.*

4.1.4. *Unipotent bundles and their 15 twists.* Let $0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0$ be a non-trivial extension. Here the space of morphisms $H^0(\text{Hom}(\mathcal{O}(-K_X), E))$ has dimension 3 and the subbundle $\mathcal{O}_X \subset E$ is responsible for this extra dimension: the space $H^0(\text{Hom}(\mathcal{O}(-K_X), \mathcal{O}_X))$ has dimension 2. There are many lifts h of the hyperelliptic involution ι since there are non-trivial automorphisms on E : any other lift is, up to a sign, given by $g \circ h \circ g^{-1}$ for some $g \in \text{Aut}(E)$. But once h is fixed, we can apply Proposition 4.1 and get that there are exactly two h -invariant Tyurin subbundles L^\pm , one of them is the unique embedding $\mathcal{O}_X \hookrightarrow E$. Possibly replacing h by $-h$, we may assume $L^+ = \mathcal{O}_X$.

Let φ^+ be a non-zero element of $H^0(\text{Hom}(\mathcal{O}(-K_X), E))$ taking values in L^+ and vanishing at say $[P] + [\iota(P)]$. Let φ^- be a non-zero element of $H^0(\text{Hom}(\mathcal{O}(-K_X), E))$ taking values in L^- . Consider the divisor defined by zeroes of $\varphi^+ \wedge \varphi^-$: as an element of the linear system $|2K_X|$, it takes the form $[P] + [\iota(P)] + [Q] + [\iota(Q)]$ including the vanishing divisor of φ^+ . Since φ^- is unique up to a constant, the divisor $[Q] + [\iota(Q)]$ is an invariant of the bundle, while $[P] + [\iota(P)]$ can be chosen arbitrarily by switching to another φ^+ .

Proposition 4.7. *The divisor $[Q] + [\iota(Q)]$ characterizes the extension E : we thus get a natural identification between the space $\mathbb{P}(H^0(\text{Hom}(\mathcal{O}(K_X)))^\vee)$ parametrizing extensions and $\mathbb{P}(H^0(\text{Hom}(\mathcal{O}(K_X)))$ parametrizing those divisors $[Q] + [\iota(Q)]$.*

The bundle L^- is degenerate if, and only if, $[Q] + [\iota(Q)] = 2[w_i]$ where w_i is a Weierstrass point. In this case, $L^- = \mathcal{O}_X(-[w_i])$ (and φ^- vanishes at w_i).

Proof. The morphism φ^- defines a natural morphism

$$\text{id}|_{L^+ \oplus \varphi^-} : \mathcal{O}_X \oplus \mathcal{O}(-K_X) \rightarrow E$$

whose determinant map vanishes at $[Q] + [\iota(Q)]$. When $Q \neq \iota(Q)$, this is a positive elementary transformation on the vector bundle $\mathcal{O}_X \oplus \mathcal{O}(-K_X)$ for a parabolic structure defined over $[Q] + [\iota(Q)]$. None of the two parabolics can be contained in the total space of the destabilizing line subbundle \mathcal{O}_X , otherwise E would be unstable. Moreover, the two parabolics cannot both be contained in a same line subbundle isomorphic to $\mathcal{O}(-K_X)$, otherwise E would be decomposable. Up to automorphism of the bundle $\mathcal{O}_X \oplus \mathcal{O}(-K_X)$, there is a unique parabolic structure over $[Q] + [\iota(Q)]$ satisfying these conditions. Hence E is well determined by the divisor $[Q] + [\iota(Q)]$. This provides a natural identification as stated, outside of the 6 special bundles for which $Q = \iota(Q) = w_i$; it extends by continuity at those points.

Since E is semi-stable and indecomposable, we have $\deg(L^-) < 0$. In the degenerate case, the only possibility is that φ^- has a single zero, at say Q , and $L^- = \mathcal{O}_X(-[\iota(Q)])$. But L^- being h -invariant, $Q = \iota(Q)$ has to be a Weierstrass point, w_i say. Conversely, if $Q = w_i$, we can chose $P \neq w_i$ making sure that φ^+ doesn't vanish at w_i . The two sections φ^+ and φ^- are however colinear at w_i and there is a linear combination $\varphi = \varphi^- + \lambda\varphi^+$ vanishing at w_i . The corresponding Tyurin subbundle L of E then is isomorphic to $\mathcal{O}_X(-[w_i])$ and thus invariant under the hyperelliptic involution. Since L^\pm are the only invariant Tyurin subbundles, we have $L^- = L \simeq \mathcal{O}_X(-[w_i])$. \square

The two hyperelliptic parabolic structures associated to h are defined by these two bundles, except for the 6 special extensions E for which L^- is degenerate. Consider now another lift h' of the hyperelliptic involution, given by $h' = g \circ h \circ g^{-1}$ for some automorphism $g \in \text{Aut}(E)$. The h' -invariant Tyurin subbundles then are L^+ and $g(L^-)$

since $\text{Aut}(E)$ fixes the subbundle $L^+ \simeq \mathcal{O}_X$. Yet the \mathbb{G}_a -part of $\text{Aut}(E)$ acts transitively on the set of non-degenerate Tyurin subbundles. Therefore, *there are exactly two hyperelliptic parabolic structures on E up to automorphism.*

Remark 4.8. *In the geometric picture, the 1-parameter family of extensions $(E_t)_{t \in \mathbb{P}^1}$ of the trivial line bundle can be seen as the tangent cone to the Kummer surface after blowing up the singular point corresponding to the trivial bundle. The strict transform of the Gunning plane $\Pi_{[w_i]}$ then intersects this \mathbb{P}^1 in a unique point which is the bundle satisfying $L^- \simeq \mathcal{O}(-[w_i])$ as above.*

Let $L_0 = \mathcal{O}([w_i] + [w_j] - K_X)$ be a non-trivial 2-torsion point of $\text{Pic}^0(X)$, $i \neq j$, and consider a non-trivial extension $0 \rightarrow L_0 \rightarrow E \rightarrow L_0 \rightarrow 0$. This time, the vector space $H^0(\text{Hom}(\mathcal{O}(-K_X), E))$ has dimension 2 and generates a 1-parameter family of Tyurin subbundles. One of them is L_0 , the only one having degree 0. It is degenerate and must be invariant, say L^+ . The group $\text{Aut}(E)$ acts transitively on the remaining part of the family and, like for unipotent bundles, h fixes one of them, say L^- . The intersection $L^+ \cap L^-$ has to be $[w_i] + [w_j]$ and L^- is therefore non-degenerate and defines the parabolic structure \mathbf{p}^- .

4.1.5. *Affine bundles.* We should also consider the case of affine bundles. We have already seen that these bundles are not invariant under the hyperelliptic involution. Hence they do not arise from elements of $\mathfrak{Bun}(X/\iota)$ and our parabolic structures \mathbf{p}^\pm are not defined. Yet Tyurin's construction naturally includes this type of bundle. Indeed, even if the notion of invariant line subbundles does not make sense here, we can of course consider the space of Tyurin subbundles of an affine bundle. Let $L_0 = \mathcal{O}_X([P] + [Q] - K_X) = \mathcal{O}_X(K_X - [\iota(P)] - [\iota(Q)])$ be a degree 0 line bundle such that $L_0^{\otimes 2} \neq \mathcal{O}_X$ and let E be the unique non-trivial extension

$$0 \longrightarrow L_0 \longrightarrow E \longrightarrow L_0^{-1} \longrightarrow 0.$$

Then $h^0(\text{Hom}(\mathcal{O}_X(-K_X), E)) = 2$. Moreover, we have $h^0(\text{Hom}(\mathcal{O}_X(-K_X), L_0)) = 1$. In other words, E possesses a 1-parameter family of Tyurin subbundles. Precisely three of them are degenerated: L_0 , a unique line subbundle $L_P \simeq \mathcal{O}_X(-P)$ of E and a unique line subbundle $L_Q \simeq \mathcal{O}_X(-Q)$ of E . They define a parabolic structure on E over the Tyurin divisor $D_E^T = [P] + [Q] + [\iota(P)] + [\iota(Q)]$ (the parabolics over $\iota(P)$ and $\iota(Q)$ are both given by L_0) and the four negative elementary transformations on E defined by these parabolics yield $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$.

4.1.6. *The 6 + 10 Gunning bundles.* Let $\vartheta \in \text{Pic}^1(X)$ be a theta characteristic and E_ϑ be the associated Gunning bundle. The subbundle $\vartheta \subset E_\vartheta$ is the unique one having degree > -1 ; it is a degenerate h -invariant Tyurin subbundle.

When ϑ is an even theta characteristic $\vartheta = [w_i] + [w_j] + [w_k] - K_X$, we have

$$h^0(\text{Hom}(\mathcal{O}_X(-K_X), \vartheta)) = h^0(\text{Hom}(\mathcal{O}_X(-K_X), E_\vartheta)) = 2$$

and all morphisms $\varphi : \mathcal{O}_X(-K_X) \rightarrow E_\vartheta$ factor through the subbundle $\vartheta \subset E_\vartheta$: there is a unique Tyurin bundle in this case. Through the identification

$$\text{Hom}(\mathcal{O}(-K_X), \vartheta) \simeq \mathcal{O}_X([w_i] + [w_j] + [w_k]),$$

the space of global sections of the sheaf of morphisms is generated by

$$1, \frac{(x - x_l)(x - x_m)(x - x_n)}{y} \in H^0(X, \mathcal{O}([w_i] + [w_j] + [w_k])),$$

where $\{i, j, k, l, m, n\} = \{0, 1, r, s, t, \infty\}$ and $w_i = (x_i, 0) \in X$. The hyperelliptic involution acts as id on the first one and $-\text{id}$ on the second one. There are two types of hyperelliptic parabolic structures on E_ϑ :

- parabolics corresponding to w_i, w_j and w_k lying in $\vartheta \hookrightarrow E_\vartheta$, the others outside;
- parabolics corresponding to w_l, w_m and w_n lying in $\vartheta \hookrightarrow E_\vartheta$, the others outside.

This implies that *up to automorphism, there are exactly two parabolic structures on a Gunning bundle E_ϑ with even theta characteristic.*

Let us now consider the case where ϑ is an odd theta characteristic $\vartheta = \mathcal{O}_X([w])$. The h -invariant Tyurin subbundles L^+ and L^- are distinct and one of them is the maximal subbundle of E_ϑ , say $L^+ = \vartheta$, which is the only degenerate Tyurin subbundle of E_ϑ . For any $P \in X$ we can choose a holomorphic section φ^+ of the line subbundle $L^+ \otimes \mathcal{O}_X(K_X)$ of $E_\vartheta \otimes \mathcal{O}_X(K_X)$ such that $\text{div}_0(\varphi^+) = [w] + [P] + [\iota(P)]$, whereas any holomorphic section φ^- of $L^- \otimes \mathcal{O}_X(K_X)$ is nowhere vanishing. Moreover, the fibres of the corresponding line subbundles of E_ϑ are colinear only over the point w . In other words, the Tyurin divisor of E_ϑ is well-defined only after the choice of a section of the destabilizing line subbundle. The birational map $\varphi^+ \oplus \varphi^- : \mathcal{O}(-K_X) \oplus \mathcal{O}(-K_X) \rightarrow E_\vartheta$ then decomposes as four successive positive elementary transformations with (Tyurin)-parabolics given by the fibres of L^- and its strict transform over $[w], [w], [P]$ and $[\iota(P)]$. The hyperelliptic parabolic \mathbf{p}_i^- on the other hand is defined by $L^+|_w = L^-|_w$ and \mathbf{p}_i^+ is elsewhere. Since $\text{Aut}(E_\vartheta)$ fixes L^+ and acts transitively on the set of line subbundles of the form $\mathcal{O}(-K_X)$, *there are, up to automorphism, exactly two parabolic structures on a Gunning bundle E_ϑ with odd theta characteristic.*

4.2. Extensions of the canonical bundle. Here, we recall some results obtained by Bertram in [7], completed in the genus 2 case by Bolognesi in [14, 15] (see also [39]).

The space of non trivial extensions $0 \rightarrow \mathcal{O}(-K_X) \rightarrow E \rightarrow \mathcal{O}(K_X) \rightarrow 0$ is $\mathbb{P}H^1(-2K_X)$ which identifies, by Serre duality, to $\mathbb{P}H^0(3K_X)^\vee$. This space naturally parametrizes the moduli space of those pairs (E, L) where $L \subset E$ is a non-degenerate Tyurin bundle. The hyperelliptic involution ι acts naturally on $H^0(3K_X)$ and thus on its dual: the invariant subspace is an hyperplane $\mathbb{P}_B^3 \subset \mathbb{P}H^0(3K_X)^\vee \simeq \mathbb{P}^4$ that naturally parametrizes those pairs (E, L) that are invariant under the involution. As we have seen in Section 4.1, most stable bundles E admit exactly two invariant and non-degenerate Tyurin subbundles and most decomposable bundles E admit only one. This suggests that \mathbb{P}_B^3 is a birational model for the 2-fold cover of \mathbb{P}_{NR}^3 ramified over the Kummer surface.

A cubic differential $\omega \in H^0(3K_X)$ writes $\omega = (a_0 + a_1x + a_2x^2 + a_3x^3 + a_4y) \left(\frac{dx}{y}\right)^{\otimes 3}$ uniquely so that the coefficients a_i provide a full set of coordinates. Let $(b_0 : b_1 : b_2 : b_3 : b_4)$ be dual homogeneous coordinates for $\mathbb{P}_B^4 := \mathbb{P}H^0(3K_X)^\vee$. We have the following description (see introductions of [7, 39] and §5 of [14])

The locus of unstable bundles is given by the natural embedding of the curve X :

$$X \hookrightarrow \mathbb{P}_B^4; (x, y) \mapsto (1 : x : x^2 : x^3 : y).$$

The locus of strictly semi-stable bundles is given by the quartic hypersurface $\text{Wed} \subset \mathbb{P}_B^4$ spanned by the 2-secant lines of X . The natural action of the hyperelliptic involution $\iota : X \rightarrow X$ on cubic differentials induces an involution on \mathbb{P}_B^4 that fixes the hyperplane $\mathbb{P}_B^3 = \{b_4 = 0\}$ and the point $(0 : 0 : 0 : 0 : 1)$.

The Narasimhan-Ramanan moduli map

$$\mathbb{P}_B^4 \dashrightarrow \mathbb{P}_{\text{NR}}^3$$

is given by the full linear system of quadrics that contain X ; it restricts to \mathbb{P}_B^3 as the full linear system of quadrics (of \mathbb{P}_B^3) that contain the six points $X \cap \mathbb{P}_B^3$. After blowing-up the locus X of unstable bundles, we get a morphism

$$\tilde{\mathbb{P}}_B^4 \rightarrow \mathbb{P}_{\text{NR}}^3$$

namely a conic bundle; its restriction to the strict transform $\tilde{\mathbb{P}}_B^3$ of \mathbb{P}_B^3 is generically $2 : 1$, ramifying over the Kummer surface $\text{Kum} \subset \mathbb{P}_{\text{NR}}^3$. The quartic hypersurface Wed restricts to \mathbb{P}_B^3 as the (dual) Weddle surface; it is sent onto the Kummer surface.

There is a Poincaré vector bundle $\mathcal{E} \rightarrow X \times \mathbb{P}_B^4$ realizing the classifying map above. Hence by restriction, there is a Poincaré bundle $\mathcal{E} \rightarrow X \times \mathbb{P}_B^3$ on the double cover \mathbb{P}_B^3 of \mathbb{P}_{NR}^3 . The projectivized Poincaré bundle $\mathbf{P}(\mathcal{E}) \rightarrow X \times \mathbb{P}_B^3$ defines a conic bundle $\mathcal{C} \rightarrow X \times \mathbb{P}_{\text{NR}}^3$ over the quotient \mathbb{P}_{NR}^3 . For each vector bundle $E \in \mathbb{P}_{\text{NR}}^3$, the fibre \mathcal{C}_E of the conic bundle represents the family of Tyurin-subbundles of E . Yet the conic bundle \mathcal{C} is not a projectivized vector bundle over \mathbb{P}_{NR}^3 , not even up to birational equivalency, because a Poincaré bundle over a Zariski-open set of \mathbb{P}_{NR}^3 does not exist [50].

4.3. Tyurin parametrization. Let E be a flat rank two vector bundle with trivial determinant bundle over X . It follows from Corollary 4.5 that, when E is stable and off the odd Gunning planes, then E can be deduced from $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$ by applying 4 positive elementary transformations, namely over the Tyurin divisor D_E^T . In fact, if we allow non reduced divisors, then this remains true for all flat bundles except even Gunning bundles. Indeed, it follows from Proposition 4.1 that we have a non degenerate map

$$\varphi^+ \oplus \varphi^- : \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X) \rightarrow E$$

by selecting φ^+ and φ^- generating H^+ and H^- respectively; non degenerate means that the image spans the generic fiber. Comparing the degree of both vector bundles, we promptly deduce that this map decomposes into 4 successive positive elementary transformations, possibly over non distinct points (this happens when the divisor $D_E^T \in |2K_X|$ is non reduced).

Conversely, let us consider a divisor, say reduced for simplicity:

$$D = [\underline{P}_1] + [\iota(\underline{P}_1)] + [\underline{P}_2] + [\iota(\underline{P}_2)] \in |2K_X|,$$

and consider also a parabolic structure \mathbf{q} over D on the trivial bundle $E_0 \rightarrow X$: given e_1 and e_2 two independent sections of E_0 , the parabolic structure is defined by

$$(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)}) \in (\mathbb{P}^1)^4$$

where $e_1 + \lambda_{\underline{P}_i} e_2$ generates the parabolic direction over \underline{P}_i , and similarly for $\iota(\underline{P}_i)$. From this data, one can associate a vector bundle with trivial determinant E by

$$(15) \quad \mathcal{O}(-K_X) \otimes \text{elm}_D^+(E_0, \mathbf{q}) \rightarrow E.$$

Table 4 lists all types of vector bundles E that can be obtained in that way.

Remark 4.9. *This list is mostly a summary of the case by case study in Section 4.1. Reasoning on the possible preimages of the destabilizing subbundle, it is straightforward to check that the above mentioned decomposable bundles are the only possible ones. Even Gunning bundles cannot be obtained: otherwise two distinct trivial subbundles of the trivial bundle would generate two distinct Tyurin subbundles on an even Gunning bundle.*

Table 4: List of vector bundles that can be obtained from Tyurin's construction.

bundle type		Tyurin divisor	reduced	parabolic structure
stable	off $\Pi_{[w_i]}$	D_E^T	yes	generic
	on $\Pi_{[w_i]}$, off $\Pi_{[w_j]}$	$2[w_i] + [P] + [\iota(P)]$	no	$(\lambda_{w_i}, \lambda_P, \lambda_{\iota(P)}) = (0, 1, \infty)$ (but P is free on $X \setminus W$)
	on $\Pi_{[w_i]} \cap \Pi_{[w_j]}$	$2[w_i] + 2[w_j]$	no	$\lambda_{w_i} \neq \lambda_{w_j}$
unipotent	generic	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_P = \lambda_{\iota(P)}$ (but Q is free)
	special	$[P] + [\iota(P)] + 2[w]$	no	$\lambda_P = \lambda_{\iota(P)}$
	twisted by $\mathcal{O}_X([w_i] - [w_j])$	$2[w_i] + 2[w_j]$	no	$\lambda_{w_i} = \lambda_{w_j}$
affine $L_0 \rightarrow E \rightarrow L_0^{-1}$	$L_0^{\otimes 2} \neq \mathcal{O}_X$ $L_0 = \mathcal{O}_X([P] + [Q] - K_X)$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$(\lambda_P, \lambda_{\iota(P)}, \lambda_Q, \lambda_{\iota(Q)})$ $= (0, 1, 0, \infty)$
semi-stable decomposable $L_0 \oplus L_0^{-1}$	generic: $L_0^{\otimes 2} \neq \mathcal{O}_X$ $L_0 = \mathcal{O}_X([P] + [Q] - K_X)$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_P = \lambda_Q \neq \lambda_{\iota(P)} = \lambda_{\iota(Q)}$
	trivial: $L_0 = \mathcal{O}_X$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_P = \lambda_{\iota(P)} \neq \lambda_Q = \lambda_{\iota(Q)}$
	twist: $L_0 = \mathcal{O}_X([w_i] - [w_j])$	$2[w_i] + 2[w_j]$	no	$\lambda_{w_i} \neq \lambda_{w_j}$
unstable decomposable $L \oplus L^{-1}$	$L = \mathcal{O}_X([P]), P \notin W$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_{\iota(P)} \neq \lambda_P = \lambda_Q = \lambda_{\iota(Q)}$
	$L = \mathcal{O}_X([w])$	$2[w] + [Q] + [\iota(Q)]$	no	$\lim_{P \rightarrow w}$ of the previous one
	$L = \mathcal{O}_X(K_X)$	$[P] + [\iota(P)] + [Q] + [\iota(Q)]$	yes	$\lambda_P = \lambda_{\iota(P)} = \lambda_Q = \lambda_{\iota(Q)}$
odd Gunning bundle	E_w	$2[w] + [P] + [\iota(P)]$ (P arbitrary)	no	$\lambda_w = \lambda_P = \lambda_{\iota(P)}$

As a consequence, the moduli space \mathcal{M}_{NR} is birational to the moduli space of parabolic structures over D on E_0 , when D runs over the linear system $|2K_X|$. Let us be more precise. Consider the parameter space

$$(\underline{P}_1, \underline{P}_2, \lambda) \in X \times X \times \mathbb{P}^1$$

and associate to each such data the parabolic structure defined on the vector bundle $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$ by

$$(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)}) := \left(\lambda, -\lambda, \frac{1}{\lambda}, -\frac{1}{\lambda} \right).$$

Equivalently, one can view the parabolic structure as the collection of points

$$(\underline{P}_1, \lambda), \quad (\iota(\underline{P}_1), -\lambda), \quad \left(\underline{P}_2, \frac{1}{\lambda} \right) \quad \text{and} \quad \left(\iota(\underline{P}_2), -\frac{1}{\lambda} \right)$$

on the total space $X \times \mathbb{P}^1$ of the projectivized \mathbb{P}^1 -bundle $\mathbb{P}(\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X))$. The natural rational map $X \times X \times \mathbb{P}^1 \dashrightarrow \mathbb{P}_{\text{NR}}^3$ is not birational however, since for a given bundle over X there are several possibilities to choose \underline{P}_1 , \underline{P}_2 and λ . One can first independently permute $\underline{P}_1 \leftrightarrow \iota(\underline{P}_1)$, $\underline{P}_2 \leftrightarrow \iota(\underline{P}_2)$ and $\underline{P}_1 \leftrightarrow \underline{P}_2$: this generates a order 8 group of permutations. Moreover, once \underline{P}_1 and \underline{P}_2 have been chosen to parametrize the linear system $|2K_X|$, there is still a freedom in the choice of λ : our choice of normalization, characterized by

$$\lambda_{\underline{P}_1} + \lambda_{\iota(\underline{P}_1)} = \lambda_{\underline{P}_2} + \lambda_{\iota(\underline{P}_2)} = 0 \quad \text{and} \quad \lambda_{\underline{P}_1} \cdot \lambda_{\underline{P}_2} = 1,$$

is invariant under the Klein 4 group $\langle z \mapsto -z, z \mapsto \frac{1}{z} \rangle$ acting on the projective variable $e_1 + ze_2$. The transformation group taking into account all this freedom is generated by the following 4 transformations

$$\begin{aligned} (\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) &\longrightarrow (\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) \\ ((\underline{P}_1, \underline{P}_2, \lambda), ((x, y), z)) &\begin{cases} \xrightarrow{\sigma_{12}} & ((\underline{P}_2, \underline{P}_1, \frac{1}{\lambda}), ((x, y), z)) \\ \xrightarrow{\sigma_\iota} & ((\iota(\underline{P}_1), \iota(\underline{P}_2), -\lambda), ((x, y), z)) \\ \xrightarrow{\sigma_{iz}} & ((\underline{P}_1, \iota(\underline{P}_2), i\lambda), ((x, y), iz)) \\ \xrightarrow{\sigma_{1/z}} & ((\underline{P}_1, \underline{P}_2, \frac{1}{\lambda}), ((x, y), \frac{1}{z})) \end{cases} \end{aligned}$$

(here, $i = \sqrt{-1}$). In fact, our choice of normalization for $(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)})$ may not be the most naive one, which would have consisted to fix 3 of them to 0, 1 and ∞ ; but our choice has the advantage that the transformation

$$\begin{aligned} (\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) &\longrightarrow (\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1) \\ ((\underline{P}_1, \underline{P}_2, \lambda), ((x, y), z)) &\longmapsto ((\underline{P}_1, \underline{P}_2, \lambda), ((x, -y), -z)) \end{aligned}$$

preserves the parabolic structure, and corresponds to the projectivized hyperelliptic involution $h : E \rightarrow \iota^* E$. In particular, the subbundles $z = 0$ and $z = \infty$ generated respectively by e_1 and e_2 precisely correspond to the two ι -invariant Tyurin subbundles of E .

The 32-order group $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}, \sigma_{1/z} \rangle$ acts faithfully on the parameter space $\underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1$. Setting $\underline{P}_1 = (\underline{x}_1, \underline{y}_1)$ and $\underline{P}_2 = (\underline{x}_2, \underline{y}_2)$, the field of rational invariant functions is generated by

$$\underline{s} := \underline{x}_1 + \underline{x}_2, \quad \underline{p} := \underline{x}_1 \underline{x}_2 \quad \text{and} \quad \underline{\lambda} := \left(\lambda^2 + \frac{1}{\lambda^2} \right) \underline{y}_1 \underline{y}_2$$

so that a quotient map (up to birational equivalence) is given by

$$(16) \quad \begin{array}{ccc} \underline{X}_1 \times \underline{X}_2 \times \mathbb{P}_\lambda^1 & \xrightarrow{(32:1)} & \mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \\ \left((\underline{x}_1, \underline{y}_1), (\underline{x}_2, \underline{y}_2), \lambda \right) & \mapsto & \left((1 : -\underline{x}_1 - \underline{x}_2 : \underline{x}_1 \underline{x}_2), (\lambda^2 + \frac{1}{\lambda^2}) \underline{y}_1 \underline{y}_2 \right) \end{array}$$

Here, $\mathbb{P}_D^2 = |2K_X|$ is just the linear system parametrizing those divisors D_E^T . This quotient is our sharp Tyurin configuration space, and we get a natural birational map

$$\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \dashrightarrow \mathbb{P}_{\text{NR}}^3$$

which can be explicitly described as follows.

Proposition 4.10. *The natural classifying map $\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \dashrightarrow \mathbb{P}_{\text{NR}}^3$ writes*

$$\begin{aligned} (\underline{s}, \underline{p}, \lambda) &\mapsto (v_0 : v_1 : v_2 : v_3) \\ &= \left(\frac{\lambda - \underline{s}\underline{p}^2 + 2(1+\sigma_1)\underline{p}^2 - (\sigma_1+\sigma_2)\underline{s}\underline{p} + (\sigma_2+\sigma_3)(\underline{s}^2 - 2\underline{p}) - \sigma_3\underline{s}}{\underline{s}^2 - 4\underline{p}} : \underline{p} : -\underline{s} : 1 \right). \end{aligned}$$

Before proving it, let us make some observations. First, the fibration $\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1 \rightarrow \mathbb{P}_D^2$ is sent onto the pencil of lines of \mathbb{P}_{NR}^3 passing through the trivial bundle $E_0 : (1 : 0 : 0 : 0)$. In fact, the surface $\{\lambda = \infty\}$ in Tyurin parameter space, corresponding to $\lambda = 0$ or ∞ , is the locus of the trivial bundle. Also, the surface defined by $\lambda = \{1, -1, i, -i\}$ corresponds to generic decomposable flat bundles and is sent onto the Kummer surface; we note that it is also defined by $\lambda^2 = 4(y_1 y_2)^2$ which, after expansion, writes

$$\begin{aligned} \lambda^2 &= \underline{p}(\underline{p} - \underline{s} + 1) \cdot \left(\underline{p}^3 - \sigma_1 \underline{p}^2 \underline{s} + \sigma_2 \underline{p} \underline{s}^2 - \sigma_3 \underline{s}^3 + (\sigma_1^2 - 2\sigma_2) \underline{p}^2 + (3\sigma_3 - \sigma_1 \sigma_2) \underline{p} \underline{s} \right. \\ &\quad \left. + \sigma_1 \sigma_3 \underline{s}^2 + (\sigma_2^2 - 2\sigma_1 \sigma_3) \underline{p} - \sigma_2 \sigma_3 \underline{s} + \sigma_3^2 \right) \end{aligned}$$

which allow us to retrieve the equation of $\text{Kum}(X) \subset \mathbb{P}_{\text{NR}}^3$.

Proof. Assume we are given $(\underline{P}_1, \underline{P}_2, \lambda)$ and the associated parabolic structure on $(E_0, \mathbf{q}) \rightarrow (X, D_E^T)$; then, we want to compute the Narasimhan-Ramanan divisor $D_E \subset \text{Pic}^1(X)$ for the corresponding vector bundle E obtained after 4 elementary transformations. Given a degree 3 line bundle L_0 , we can look at holomorphic sections $s_0 : X \rightarrow E_0 \otimes L_0$; it is straightforward to check that a section $s_1 e_1 + s_2 e_2$ taking value in Tyurin parabolic directions over D_E^T will produce, after elementary transformations, a holomorphic section of $E \otimes L_0(K_X - D_E^T)$, showing that $L_0(K_X - D_E^T) = L_0(-K_X) \in D_E$. Since sections of $L_0 = \mathcal{O}_X([P_1] + [P_2] + [\infty])$ are generated by $\langle 1, \frac{y+y_1}{x-x_1} - \frac{y+y_2}{x-x_2} \rangle$, up to automorphisms of E_0 , we can assume $s_1 = 1$ and $s_2 = f := \frac{y+y_1}{x-x_1} - \frac{y+y_2}{x-x_2}$. Therefore, computing the cross-ratio, we get

$$\gamma := \frac{\lambda_{\underline{P}_2} - \lambda_{\underline{P}_1}}{\lambda_{\iota(\underline{P}_1)} - \lambda_{\underline{P}_1}} : \frac{\lambda_{\underline{P}_2} - \lambda_{\iota(\underline{P}_2)}}{\lambda_{\iota(\underline{P}_1)} - \lambda_{\iota(\underline{P}_2)}} = \frac{f(\underline{P}_2) - f(\underline{P}_1)}{f(\iota(\underline{P}_1)) - f(\underline{P}_1)} : \frac{f(\underline{P}_2) - f(\iota(\underline{P}_2))}{f(\iota(\underline{P}_1)) - f(\iota(\underline{P}_2))}$$

which, after reduction, gives

$$\begin{aligned} \frac{4\underline{y}_1 \underline{y}_2 \gamma}{(\underline{x}_1 - \underline{x}_2)^2} &= (-\text{Diag}(\underline{P}_1, \underline{P}_2) \cdot 1 + \text{Prod}(\underline{P}_1, \underline{P}_2) \cdot \text{Sum} \\ &\quad - \text{Sum}(\underline{P}_1, \underline{P}_2) \cdot \text{Prod} + \text{Diag}) \end{aligned}$$

with notations of Section 3.6. On the other hand, from Tyurin parameters, we get

$$\gamma = -\frac{(1 - \lambda^2)^2}{4\lambda^2}$$

hence the result. \square

The total space $(X_1 \times X_2 \times \mathbb{P}_\lambda^1) \times (X \times \mathbb{P}_z^1)$ is equipped with the 4 rational sections

$$(\underline{P}_1, \lambda), (\iota(\underline{P}_1), -\lambda), \left(\underline{P}_2, \frac{1}{\lambda}\right), \left(\iota(\underline{P}_2), -\frac{1}{\lambda}\right) : (X_1 \times X_2 \times \mathbb{P}_\lambda^1) \rightarrow (X \times \mathbb{P}_z^1)$$

which are globally invariant under the action of $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}, \sigma_{1/z} \rangle$. The quotient provides a projective Poincaré bundle, namely a (non trivial) \mathbb{P}^1 -bundle over $(\mathbb{P}_D^2 \times \mathbb{P}_\lambda^1) \times X$ (actually, over an open set of the parameters) equipped with a universal parabolic structure. After positive elementary transformation, we get a universal \mathbb{P}^1 -bundle over an open subset of \mathbb{P}_{NR}^3 . However, we cannot lift the construction to a vector bundle because the action of $\langle z \mapsto -z, z \mapsto \frac{1}{z} \rangle$ (induced by $\langle \sigma_{iz}^2, \sigma_{1/z} \rangle$) does not lift to a linear GL_2 -action (indeed, $\begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}$ and $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ do not commute). This is the reason why there is no Poincaré bundle for \mathbb{P}_{NR}^3 , but only a projective version of it. The ambiguity is killed-out if we do not take $\sigma_{1/z}$ into account, meaning that we choose one of the two h -invariants Tyurin subbundles: we then obtain Bolognesi's Poincaré bundle mentioned in Section 4.2, which here is explicitly given as follows. Consider the vector bundle

$$\tilde{\mathcal{E}} = p^*(\mathcal{O}_X(K_X)) \otimes \text{elm}_{\delta_1, \delta_2, \delta_3, \delta_4}^+((X_1 \times X_2 \times \mathbb{P}^1) \times (X \times \mathbb{C}^2))$$

over $(X_1 \times X_2 \times \mathbb{P}^1) \times X$, where $\delta_1 : p = p_1$, $\delta_2 : p = \iota(p_1)$, $\delta_3 : p = p_2$, $\delta_4 : p = \iota(p_2)$ if p denotes the projection from $X_1 \times X_2 \times \mathbb{P}^1 \times X$ to X and p_i the projection to X_i ; and the parabolic structure over these divisors is given respectively by

$$(\underline{P}_1, \underline{P}_2, \lambda, \underline{P}_1, \begin{pmatrix} \lambda \\ 1 \end{pmatrix}), (\underline{P}_1, \underline{P}_2, \lambda, \iota(\underline{P}_1), \begin{pmatrix} -\lambda \\ 1 \end{pmatrix}), (\underline{P}_1, \underline{P}_2, \lambda, \underline{P}_2, \begin{pmatrix} 1 \\ \lambda \end{pmatrix}), (\underline{P}_1, \underline{P}_2, \lambda, \iota(\underline{P}_2), \begin{pmatrix} 1 \\ -\lambda \end{pmatrix}).$$

This vector bundle is clearly invariant for the action

$$(\underline{P}_1, \underline{P}_2, \lambda, P, Z) \begin{cases} \xrightarrow{\sigma_{12}} & (\underline{P}_2, \underline{P}_1, \frac{1}{\lambda}, P, Z) \\ \xrightarrow{\sigma_\iota} & (\iota(\underline{P}_1), \iota(\underline{P}_2), -\lambda, P, Z) \\ \xrightarrow{\sigma_{iz}} & \left((\underline{P}_1, \iota(\underline{P}_2), i\lambda, P, \begin{pmatrix} \sqrt{i} & 0 \\ 0 & \frac{1}{\sqrt{i}} \end{pmatrix} Z) \right), \end{cases}$$

i.e. $\tilde{\mathcal{E}} \simeq \sigma^* \tilde{\mathcal{E}}$ for each $\sigma \in \langle \sigma_{12}, \sigma_\iota, \sigma_{iz} \rangle$. The quotient (in the sense of [9]) thus defines a universal vector bundle $\mathcal{E} \rightarrow X \times B$ with trivial determinant bundle parametrized by the 2-cover $B = (X_1 \times X_2 \times \mathbb{P}^1) / \langle \sigma_{12}, \sigma_\iota, \sigma_{iz} \rangle = \mathbb{P}_D^2 \times \mathbb{P}_\lambda^1$ of an open set of \mathcal{M}_{NR} .

5. FLAT PARABOLIC VECTOR BUNDLES OVER THE QUOTIENT X/ι

The aim of this section is to completely describe the space $\mathfrak{Bun}(X/\iota)$ of flat parabolic vector bundles over the quotient X/ι . It can be covered by 3-dimensional projective charts patched together by birational transition maps. Our main focus will lay on the Bertram chart \mathbb{P}_B^3 (see Section 4.2). We will see that this chart has a particularly rich geometry (see Figure 3). Precisely, there is a natural embedding $X/\iota \hookrightarrow \mathbb{P}_B^3$ as a twisted cubic and $\phi|_{\mathbb{P}_B^3} : \mathbb{P}_B^3 \subset \mathfrak{Bun}(X/\iota) \xrightarrow{2:1} \mathfrak{Bun}(X)$ is defined by the linear system of quadrics passing through the 6 conic points of X/ι . The Galois involution $\Upsilon : \mathfrak{Bun}(X/\iota) \xrightarrow{\sim} \mathfrak{Bun}(X/\iota)$ of ϕ is defined by elementary transformations: $\Upsilon = \mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_W^+$. After restriction to the chart \mathbb{P}_B^3 , it is known as Geiser involution (see Dolgachev [17]); its decomposition as sequence of blow-up and contraction directly follows from the study of wall-crossing phenomena when weights vary inside $\frac{1}{6} < \mu < \frac{5}{6}$. In this picture, unipotent bundles come from the parabolic bundles parametrized by the cubic X/ι , and twisted unipotent bundles come from the 15 lines passing through 2 among 6 points. The

Gunning planes with even theta-characteristic come from the 20 planes passing through 3 among 6 points, while odd Gunning planes come from the 6 conic points of X/ι , that are indeterminacy points for $\phi|_{\mathbb{P}_B^3}$. Finally, the Kummer surface lifts as the dual Weddle surface (another quartic birational model of $\text{Kum}(X)$). These results are summarized in the Figure 3.

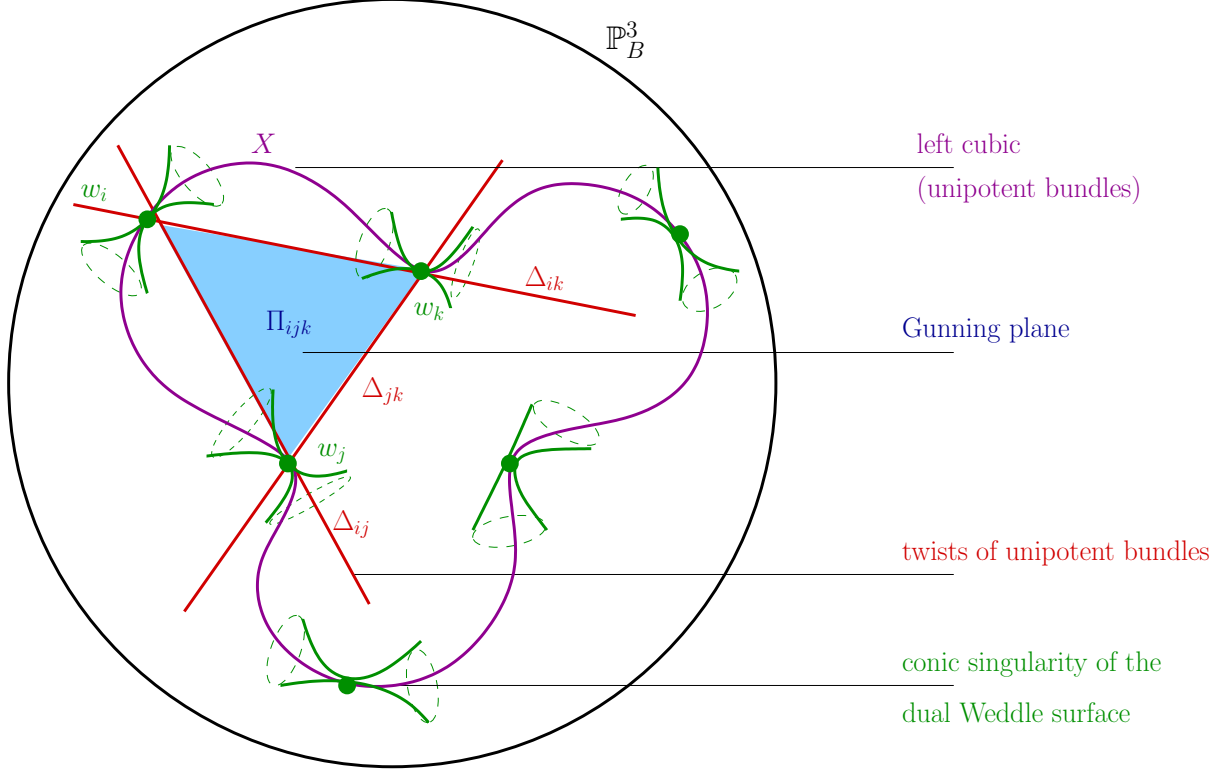


FIGURE 3. Special bundles in the chart \mathbb{P}_B^3 .

5.1. **Flatness criterion.** Consider the data $(\underline{E}, \underline{\nabla}, \underline{p})$ where

- \underline{E} is a rank 2 vector bundle over \mathbb{P}^1 ,
- $\underline{\nabla} : \underline{E} \rightarrow \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$ is a rank 2 logarithmic connection on \underline{E} with polar divisor $\underline{W} = [0] + [1] + [r] + [s] + [t] + [\infty]$ and residual eigenvalues 0 and $\frac{1}{2}$ over each pole,
- $\underline{p} = (\underline{p}_0, \underline{p}_1, \underline{p}_r, \underline{p}_s, \underline{p}_t, \underline{p}_\infty)$ the parabolic structure defined by the $\frac{1}{2}$ -eigendirections over $x = 0, 1, r, s, t, \infty$.

Via the Riemann-Hilbert correspondance, an equivalent data is the monodromy representation $\pi_1(\mathbb{P}^1 \setminus \{0, 1, r, s, t, \infty\}) \rightarrow \text{GL}_2$ with local monodromy $\sim \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ at the punctures. We denote by $\mathfrak{Con}(X/\iota)$ the coarse moduli space of such parabolic connections $(\underline{E}, \underline{\nabla}, \underline{p})$. Note that the parabolic structure \underline{p} is actually determined by the connection $(\underline{E}, \underline{\nabla})$ so that we do not need to specify it. However, it plays a crucial role in the bundle map.

We denote by $\mathfrak{Bun}(X/\iota)$ the coarse moduli space of the parabolic bundles $(\underline{E}, \underline{p})$ subjacent to some irreducible parabolic connection $(\underline{E}, \underline{\nabla}, \underline{p})$. We note that, from Fuchs

relations, we get that

$$\deg(E) = -3 \quad \text{for any } (\underline{E}, \underline{p}) \in \mathfrak{Bun}(X/\iota).$$

Following [13, 2], we have the complete characterization of flat parabolic bundles:

Proposition 5.1. *Given a parabolic bundle $(\underline{E}, \underline{p})$ like above, there exists a connection $\underline{\nabla}$ compatible with the parabolic structure like above if and only if $\deg(\underline{E}) = -3$ and*

- *either $(\underline{E}, \underline{p})$ is indecomposable,*
- *or $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ with 2 parabolics defined by the fibres over the Weierstrasspoints of the line subbundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, the 4 other ones by $\mathcal{O}_{\mathbb{P}^1}(-2)$,*
- *or $\underline{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ with all parabolics defined by $\mathcal{O}_{\mathbb{P}^1}(-3)$.*

Moreover, in each case, one can choose $\underline{\nabla}$ irreducible.

Proof. We refer to the proof of Proposition 3 in [2] to show that indecomposable parabolic bundles are flat: this part of their proof does not use genericity of eigenvalues. In the decomposable case, $\underline{E} = L_1 \oplus L_2$ and parabolics are distributed along L_1 and L_2 giving a decomposition $\underline{W} = D_1 + D_2$. If it exists, a connection $\underline{\nabla}$ writes in matrix form

$$\underline{\nabla} = \begin{pmatrix} \nabla_1 & \theta_{1,2} \\ \theta_{2,1} & \nabla_2 \end{pmatrix}$$

where

- $\nabla_i : L_i \rightarrow L_i \otimes \Omega_{\mathbb{P}^1}^1(D_i)$ is a logarithmic connection with eigenvalues $\frac{1}{2}$ for $i = 1, 2$;
- $\theta_{i,j} : L_j \rightarrow L_i \otimes \Omega_{\mathbb{P}^1}^1(D_i)$ is a morphism for $i \neq j$.

Fuchs relation for \underline{E} gives $\deg(\underline{E}) = -3$, and for ∇_i , gives

$$-2 \deg(L_i) = \text{number of parabolics lying on } L_i.$$

It follows that the only flat decomposable parabolic bundles are those listed in the statement. Now we note that connections ∇_i exist and are uniquely determined by above conditions. Setting $\theta_{i,j} = 0$, we get a (totally reducible) parabolic connection $\underline{\nabla}$ on $(\underline{E}, \underline{p})$. In all cases, $\theta_{i,j}$ are morphisms $\mathcal{O}_{\mathbb{P}^1}(n) \rightarrow \mathcal{O}_{\mathbb{P}^1}(n+1)$ for some n and live in a 2-dimensional vector space. We claim that

- $\underline{\nabla}$ is reducible if, and only if, one of the $\theta_{i,j} = 0$,
- $\underline{\nabla}$ is totally reducible if, and only if, all $\theta_{i,j} = 0$.

Indeed, if a line bundle $L \hookrightarrow \underline{E}$ is $\underline{\nabla}$ -invariant, then Fuchs relation for $\underline{\nabla}|_L$ gives the following possible cases:

- $\deg(L) = -3$ and L contains all parabolics;
- $\deg(L) = -2$ and L contains 4 parabolics;
- $\deg(L) = -1$ and L contains 2 parabolics;
- $\deg(L) = 0$ and L contains no parabolics.

This forces L to be one of direct summands of the decomposable cases above. For instance, when $\deg(L) = -3$, either $L \hookrightarrow \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ and must coincide with the second direct summand (since both must contain all parabolics), or $L \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ but then L intersects the first direct summand at only one point and thus cannot share the 2 parabolics on $\mathcal{O}_{\mathbb{P}^1}(-1)$. \square

It follows from Proposition 5.1 above that the only flat decomposable parabolic bundles are

- $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ with 2 parabolics defined by $\mathcal{O}_{\mathbb{P}^1}(-1)$, the 4 other ones by $\mathcal{O}_{\mathbb{P}^1}(-2)$, and
- $\underline{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ with all parabolics defined by $\mathcal{O}_{\mathbb{P}^1}(-3)$.

For each such bundle $(\underline{E}, \underline{p})$, the space of connections is $\mathbb{C}_{\theta_{1,2}}^2 \times \mathbb{C}_{\theta_{2,1}}^2$ (the $\theta_{i,j}$ are those defined in the proof of Proposition 5.1) where $\{0\} \times \mathbb{C}^2$ and $\mathbb{C}^2 \times \{0\}$ stand for reducible connections and $\{0\} \times \{0\}$ for the unique totally reducible one. The automorphism group of $(\underline{E}, \underline{p})$ is \mathbb{C}^* acting as follows:

$$\mathbb{C}^* \times \mathbb{C}^4 \rightarrow \mathbb{C}^4 ; (\lambda, (a_0, a_1, b_0, b_1)) \mapsto (\lambda a_0, \lambda a_1, \lambda^{-1} b_0, \lambda^{-1} b_1).$$

The GIT quotient is the affine threefold $xy = zw$ where $x = a_0 a_1$, $y = b_0 b_1$, $z = a_0 b_1$ and $w = a_1 b_0$; the singular point $x = y = z = w = 0$ stands for reducible connections.

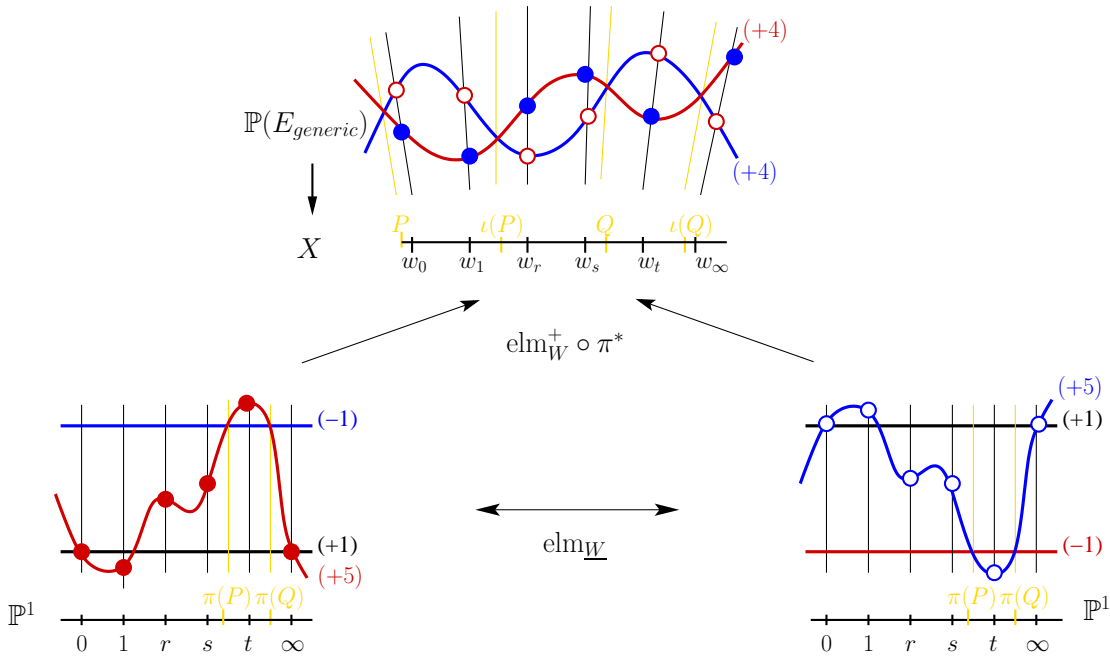


FIGURE 4. A generic stable bundle on X .

5.2. Dictionary: how special bundles on X occur as special bundles on X/ι . Let us recall the construction of the map $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$ (see Sections 2.2 and 2.2.1). Given a flat parabolic bundle $(\underline{E}, \underline{p})$ in $\mathfrak{Bun}(X/\iota)$, we lift it up to the curve X as $\pi^*(\underline{E}, \underline{p}) = (\tilde{E}, \tilde{\mathbf{p}})$, then apply elementary transformations $(E, \mathbf{p}) := \text{elm}_W^+(\tilde{E}, \tilde{\mathbf{p}})$ over the Weierstrass points and get a determinant-free vector bundle E over X , an element of $\mathfrak{Bun}(X)$. Conversely, given a generic bundle E on X , say stable and off the Gunning planes, then it has exactly two ι -invariant anti-canonical subbundles $\mathcal{O}_X(-K_X) \hookrightarrow E$ (see Corollary 4.4); consider the parabolic structure \mathbf{p} defined by the fibres over the Weierstrass points of one of them $L \subset E$. Then after applying elementary transformations over the Weierstrass points $(\tilde{E}, \tilde{\mathbf{p}}) := \text{elm}_W^-(E, \mathbf{p})$, we get the lift of a unique parabolic bundle $(\underline{E}, \underline{p})$ on X/ι ; precisely, $\tilde{E} = \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-2K_X)$ and $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. The two anti-canonical subbundles $L, L' \subset E$, being ι -invariant, descend as two subbundles of $(\underline{E}, \underline{p})$; one easily checks that they are the destabilizing

bundle $\underline{L} = \mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E} \simeq \mathcal{O}_{\mathbb{P}^1}(-1) \times \mathcal{O}_{\mathbb{P}^1}(-2)$ and the unique $\underline{L}' \simeq \mathcal{O}_{\mathbb{P}^1}(-4) \subset \underline{E}$ containing all parabolics \underline{p} .

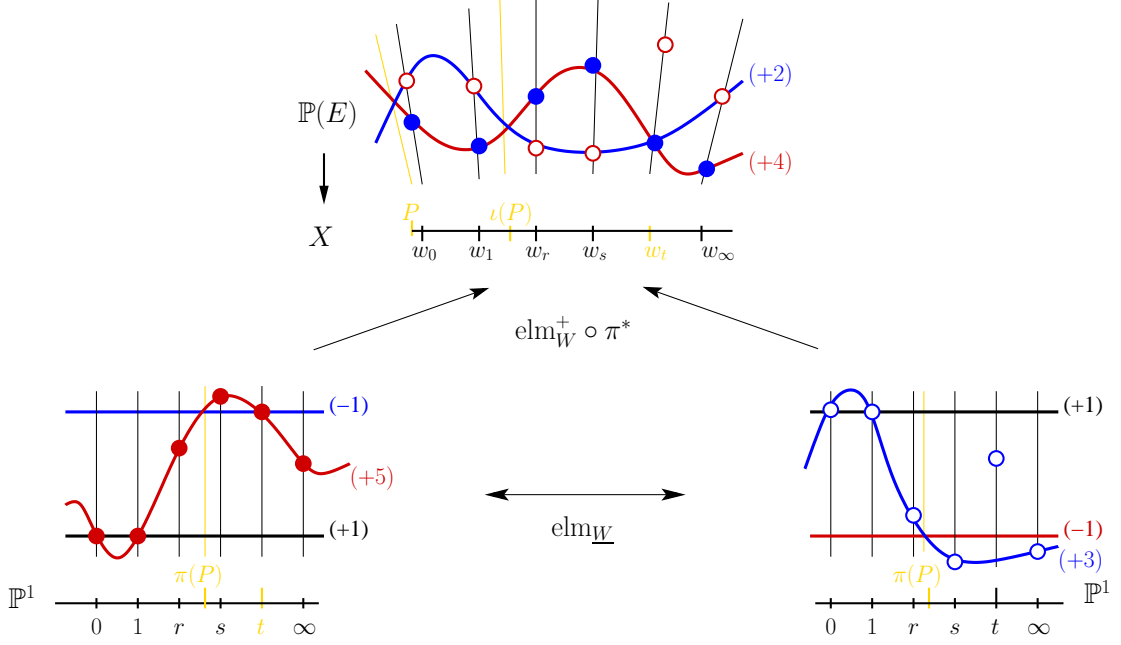
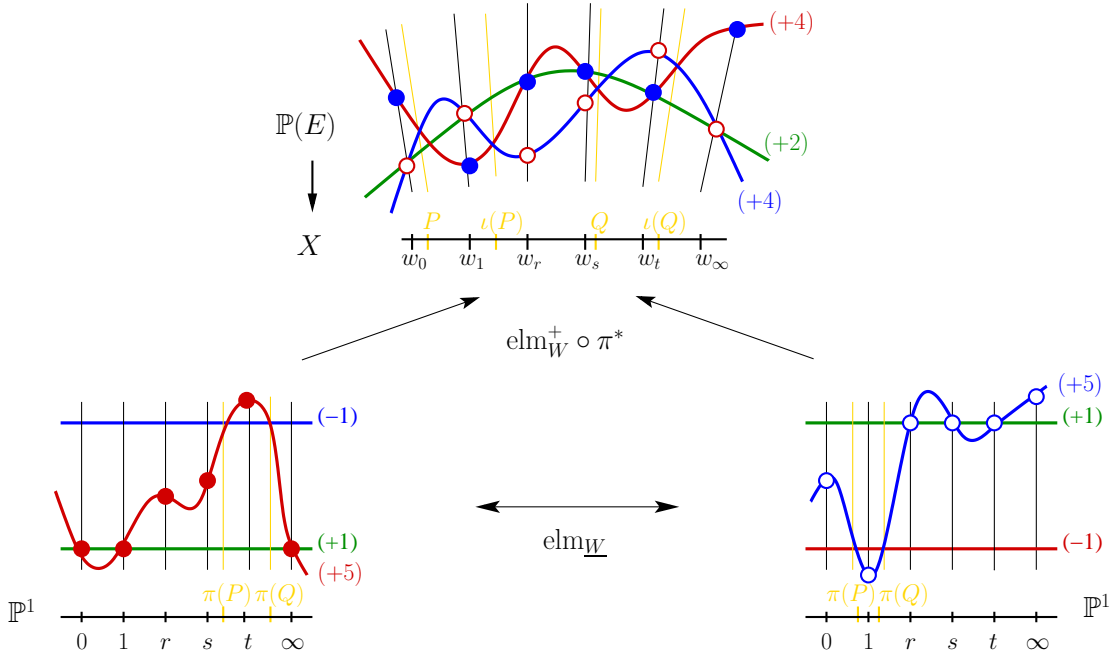


FIGURE 5. A stable bundle belonging to the odd Gunning plane $\Pi_{[w_t]}$.

In Figure 4, we can see the projectivized total space of the parabolic bundle associated to E (a ruled surface), and its two preimages \underline{E} and \underline{E}' in $\mathfrak{Bun}(X/\iota)$. The anti-canonical subbundles L and L' of E , and the corresponding subbundles of \underline{E} and \underline{E}' , are the blue and red curves (sections) on the ruled surfaces. We can see the self-intersection of the curves in each case. Parabolics are just points in Weierstrass fibers; those corresponding to \underline{p} and \underline{p} (defined by the blue curve L up-side) are the red ones and those corresponding to \underline{p}' and \underline{p}' (defined by the red curve L' up-side) are the blue ones. The intersection of the two curves determines (in each ruled surface) the Tyurin divisor D_E^T . The Galois involution of $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$ permutes the roles of L and L' ; down-side, the elementary transformation permutes the role of the two curves. The special case drawn in figure 5.2 where one of the two $+4$ -curves is reducible, we obtain a stable bundle on an odd Gunning plane. Another special case, drawn in figure 5.2 arises when $\mathbb{P}E$ possesses an invariant (but not anti-canonical) $+2$ -curve (drawn here in green) containing three parabolics of each type (red and blue). This configuration corresponds to a stable bundle on an even Gunning plane.

We now list the parabolic bundles of $\mathfrak{Bun}(X/\iota)$ giving rise to special bundles of $\mathfrak{Bun}(X)$ and illustrate on pictures the corresponding configurations of curves and points on the ruled surfaces.

5.2.1. Generic decomposable bundles. Let $E = L_0 \oplus L_0^{-1}$, where $L_0 = \mathcal{O}([P] + [Q] - K_X)$ is not 2-torsion: $L_0^2 \neq \mathcal{O}_X$. Assume also, for simplicity, that neither P , nor Q is a Weierstrass point. Recall (see Section 4.1.2) that, up to automorphism, there is a unique parabolic structure \underline{p} which is defined by the line subbundle associated to any


 FIGURE 6. A stable bundle belonging to the even Gunning plane $\Pi_{[w_r]+[w_s]-[w_t]}$.

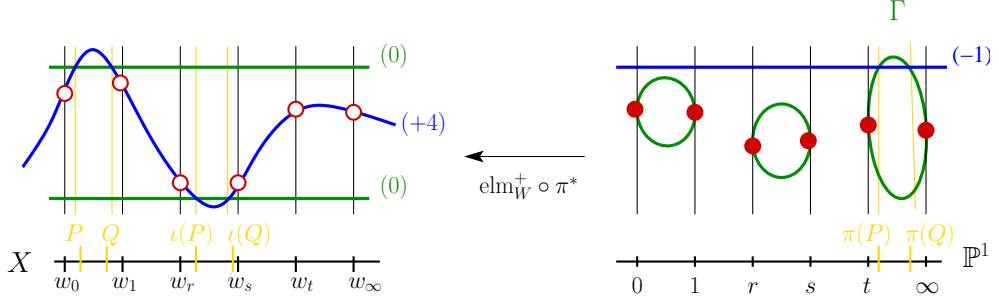
embedding $\mathcal{O}_X(-K_X) \hookrightarrow E$. On the projective bundle $\mathbb{P}E$, there are two sections $\sigma_0, \sigma_\infty : X \rightarrow \mathbb{P}E$ coming from the two direct summands L_0 and L_0^{-1} respectively, both having 0 self-intersection. They are permuted by the involution $\iota : X \rightarrow X$. On the other hand, the anticanonical embedding defines a section $\sigma : X \rightarrow \mathbb{P}E$ intersecting σ_0 at $[P] + [Q]$ and σ_∞ at $[\iota(P)] + [\iota(Q)]$. One can view $\mathbb{P}E$ as the fiber-wise compactification of $\mathcal{O}_X([P] + [Q] - [\iota(P)] - [\iota(Q)])$ with σ_0 as the zero section and σ_∞ as the compactifying section; then σ is a rational section with divisor $[P] + [Q] - [\iota(P)] - [\iota(Q)]$.

For the corresponding parabolic bundle $(\underline{E}, \underline{p})$, the anticanonical embedding descends as the destabilizing subbundle $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. On the other hand, σ_0 and σ_∞ , being permuted by the involution ι , descend as a 2-section $\Gamma \subset \mathbb{P}(\underline{E})$, thus intersecting a generic member of the ruling twice. Moreover, Γ intersects twice the section $\sigma_{-1} : \mathbb{P}^1 \rightarrow \mathbb{P}(\underline{E})$ defined by the destabilizing bundle $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E}$, namely at $\pi(P)$ and $\pi(Q)$ (where $\pi : X \rightarrow \mathbb{P}^1 = X/\iota$ is the hyperelliptic projection). The restriction of the ruling projection $\mathbb{P}(\underline{E}) \rightarrow \mathbb{P}^1$ to the curve Γ :

$$\Gamma \rightarrow \mathbb{P}^1 (= X/\iota)$$

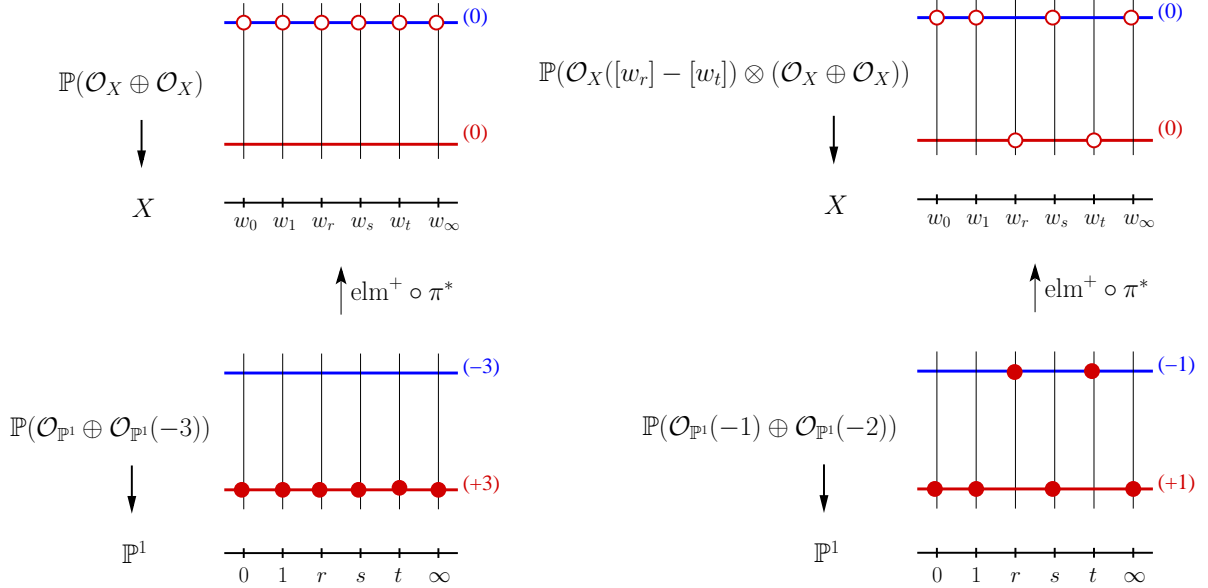
is a 2 : 1-cover branching precisely over the branching divisor \underline{W} of $\pi : X \rightarrow \mathbb{P}^1$ (orbifold points of X/ι). The parabolic structure \underline{p} is precisely located at the double points of $\Gamma \subset \mathbb{P}(\underline{E})$ over \underline{W} .

Conversely, a parabolic structure \underline{p} on $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ gives rise to a decomposable bundle E if, and only if, there is a smooth curve $\Gamma \subset \mathbb{P}(\underline{E})$ belonging to the linear system defined by $|2[\sigma_{-1}] + 4[f]|$ (with f any fiber of the ruling and σ_{-1} the negative section as before) such that Γ passes through all 6 parabolic points \underline{p} and is moreover vertical at these points (i.e. tangent to the ruling).

FIGURE 7. A generic decomposable bundle on X .

5.2.2. *The trivial bundle and its 15 twists.* Up to automorphism, the trivial bundle $E_0 = \mathcal{O}_X \oplus \mathcal{O}_X$ has a unique parabolic structure \underline{p} , which is defined by any line subbundle $\mathcal{O}_X \hookrightarrow E_0$. Descending to \mathbb{P}^1 , we get the decomposable bundle $\underline{E}_0 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ with parabolic structure \underline{p} defined by any line subbundle $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow \underline{E}_0$. Note that $(\underline{E}_0, \underline{p})$ is a fixed point of the Galois involution $\mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_W^+$. Similarly, $E_\tau = \tau \otimes E_0$ with $\tau = \mathcal{O}_X([w_i] - [w_j])$ a 2-torsion line bundle, comes from the decomposable parabolic bundle $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ having parabolics \underline{p}_i and \underline{p}_j lying in the first direct summand, the other ones in the second.

These 16 parabolic bundles are exactly the flat decomposable bundles listed in Proposition 5.1.

FIGURE 8. The trivial bundle over X and one of its twists.

5.2.3. *The unipotent family and its 15 twists.* A generic non trivial extension $0 \rightarrow \mathcal{O}_X \rightarrow E \rightarrow \mathcal{O}_X \rightarrow 0$ has two hyperelliptic parabolic structures:

- \underline{p} defined by some embedding $\mathcal{O}_X(-K_X) \hookrightarrow E$ (unique up to bundle automorphism);
- \underline{p}' defined by the destabilizing bundle $\mathcal{O}_X \hookrightarrow E$.

They respectively descend to elements of

- $\underline{\Delta} = \{(\underline{E}, \underline{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p} \subset \mathcal{O}_{\mathbb{P}^1}(-3) \subset \underline{E}\};$
- $\underline{\Delta}' = \{(\underline{E}', \underline{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3) \text{ and } \underline{p}' \subset \mathcal{O}_{\mathbb{P}^1}(-4) \subset \underline{E}'\}.$

The study of non-trivial extensions $0 \rightarrow \tau \rightarrow E \rightarrow \tau \rightarrow 0$ where $\tau = \mathcal{O}_X([w_i] - [w_j])$ is a 2-torsion line bundle, can be deduced from the study of the corresponding unipotent bundles $\tau \otimes E$ by applying $\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \text{elm}_{[w_i]+[w_j]}^+$ on \underline{E} or, equivalently, by interchanging on $\tau \otimes E$ the parabolic directions p_i and p_j with p'_i and p'_j respectively. We get a 1-parameter family $\underline{\Delta}_{i,j}$ naturally parametrized by X/ι . There are two hyperelliptic parabolic structures for such a bundle E :

- \underline{p} with parabolics p_i and p_j on $\mathcal{O}_X \hookrightarrow E$ and the others outside;
- \underline{p}' with parabolics p_i and p_j outside $\mathcal{O}_X \hookrightarrow E$ and the others on it.

They respectively descend as elements of

- $\underline{\Delta}_{i,j} = \{(\underline{E}, \underline{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-2), \forall k \neq i, j\};$
- $\underline{\Delta}'_{i,j} = \{(\underline{E}', \underline{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}'_i, \underline{p}'_j \subset \mathcal{O}_{\mathbb{P}^1}(-1)\}.$

Again, $\mathcal{O}(-3) \otimes \text{elm}_W^+$ point-wise permutes $\underline{\Delta}_{i,j}$ and $\underline{\Delta}'_{i,j}$.

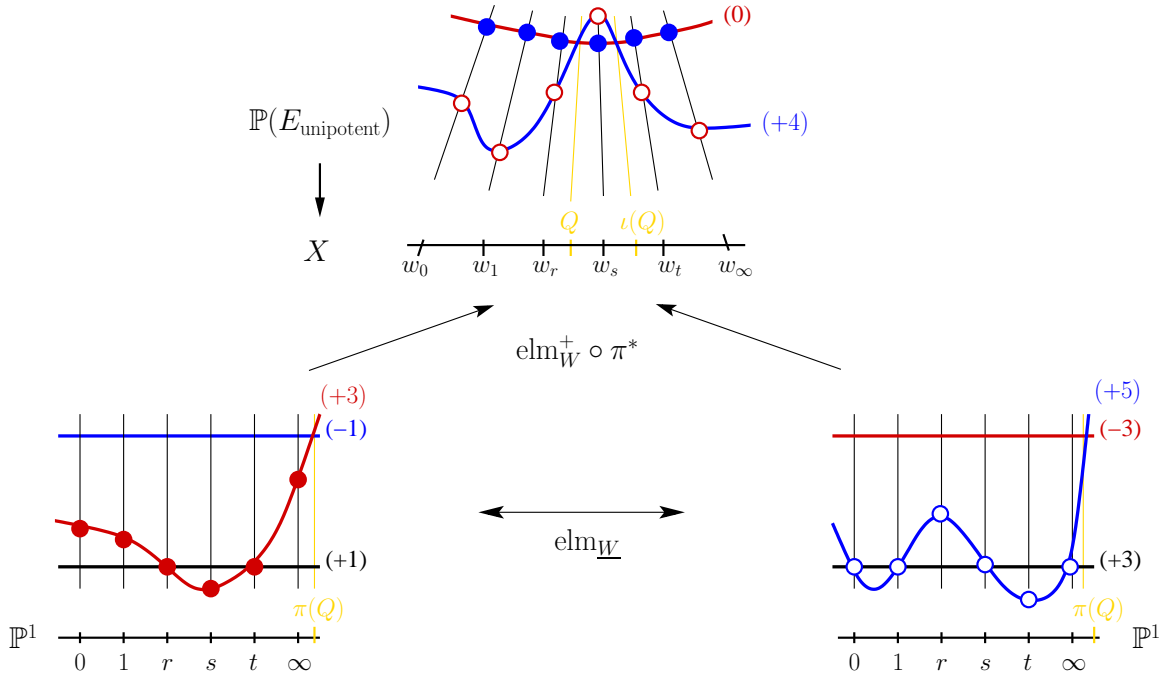
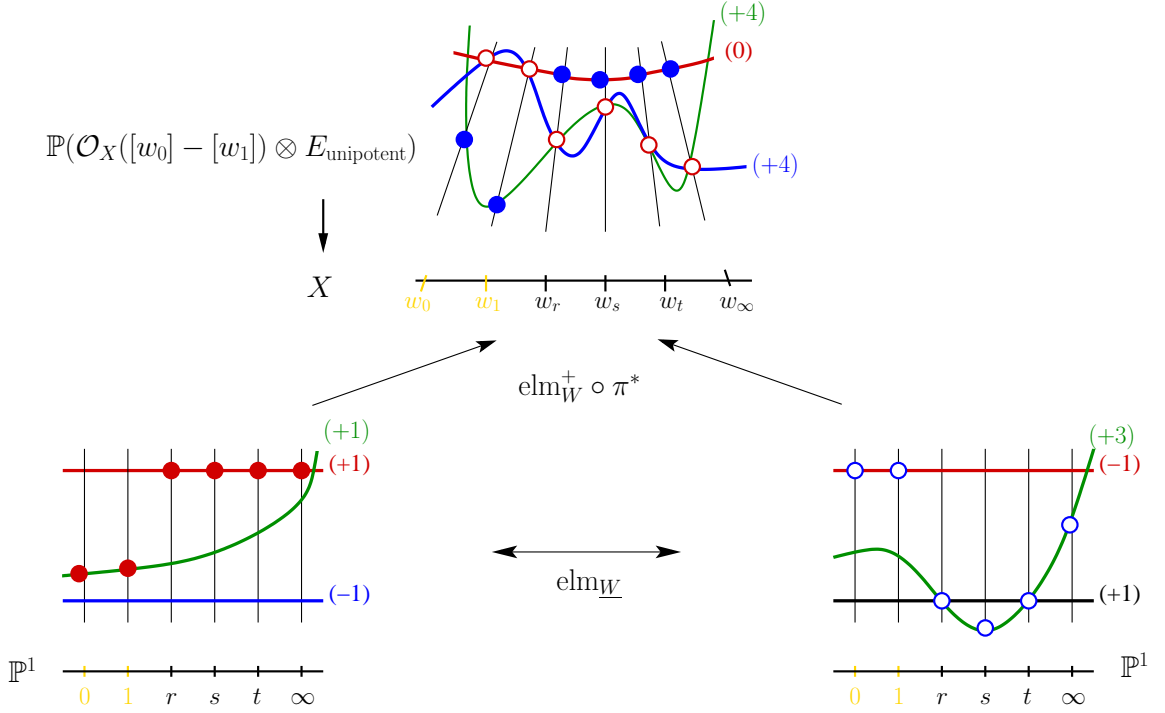


FIGURE 9. A unipotent bundle over X .

FIGURE 10. Twist of a unipotent bundle over X .

Denote by Δ the 1-parameter family of unipotent bundles in $\mathfrak{Bun}(X)$ and by $\underline{\Delta}$ and $\underline{\Delta}'$ its respective preimages on $\mathfrak{Bun}(X/\iota)$. Both of these families are naturally parametrized by our base X/ι : the extension class of $E \in \Delta$ is characterized by the intersection locus of the two special subbundles $\mathcal{O}_X(-K_X), \mathcal{O}_X \hookrightarrow E$, an element of $|\mathcal{O}_X(K_X)| \simeq |\mathcal{O}_{\mathbb{P}^1}(1)|$. Conversely, the intersection locus of two subbundles $\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$, respectively $\mathcal{O}_{\mathbb{P}^1}, \mathcal{O}_{\mathbb{P}^1}(-4) \hookrightarrow \underline{E}'$, defines an element of $|\mathcal{O}_{\mathbb{P}^1}(1)|$. This unambiguously defines isomorphisms $\Delta \simeq \underline{\Delta} \simeq \underline{\Delta}'$, the latter one being induced by $\mathcal{O}(-3) \otimes \text{elm}_W^+$. Remind (see [41]) that, despite the point-wise identification just mentioned, any point of $\underline{\Delta}$ is arbitrary close to any point of $\underline{\Delta}'$ in the sense that they can be simultaneously approximated by some deformation of stable parabolic bundles. This will give rise to a flop phenomenon when we will compare certain semi-stable projective charts. The same phenomenon occurs for twisted unipotent bundles.

5.2.4. Affine bundles. Affine bundles cannot occur from elements in $\mathfrak{Bun}(X/\iota)$ since they are not invariant under the hyperelliptic involution.

5.2.5. The 6 + 10 Gunning bundles and Gunning planes. We now list how arise the unique non trivial extensions $0 \rightarrow \mathcal{O}_X(\vartheta) \rightarrow E \rightarrow \mathcal{O}_X(-\vartheta) \rightarrow 0$ where ϑ runs over the 16 theta characteristics $\vartheta^2 = K_X$.

Six odd theta characteristics. For odd theta characteristics $\vartheta = [w_i]$ (lying on the divisor Θ) the two hyperelliptic parabolic structures are:

- p with parabolic p_i in $\mathcal{O}_X(\vartheta) \hookrightarrow E$ and the others outside;
- p' with all parabolics in $\mathcal{O}_X(\vartheta) \hookrightarrow E$ except p_i .

They respectively descend as

- $Q_i : \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and $\underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-2), \forall k \neq i$;

- $Q'_i : \underline{E}' = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ and $\underline{p}_i' \subset \mathcal{O}_{\mathbb{P}^1}$.

Analogously, the Gunning plane Π_ϑ descends as

- $\underline{\Pi}_i = \{(\underline{E}, \underline{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_i \subset \mathcal{O}_{\mathbb{P}^1}(-1)\};$
- $\underline{\Pi}'_i = \{(\underline{E}', \underline{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_k' \subset \mathcal{O}_{\mathbb{P}^1}(-3), \forall k \neq i\}.$

Ten even theta characteristics. Somehow different is the case of even theta characteristics $\vartheta = [w_i] + [w_j] - [w_k]$. Denote by $\underline{W} = \{i, j, k\} \cup \{l, m, n\}$. The two hyperelliptic parabolic structures are:

- \underline{p} with parabolics p_i, p_j and p_k in $\mathcal{O}_X(\vartheta) \hookrightarrow E$ and the others outside;
- \underline{p}' with parabolics p_l, p_m and p_n in $\mathcal{O}_X(\vartheta) \hookrightarrow E$ and the others outside.

They respectively descend as elements of

- $Q_{i,j,k} : \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and $\underline{p}_l, \underline{p}_m, \underline{p}_n \subset \mathcal{O}_{\mathbb{P}^1}(-1);$
- $Q_{l,m,n} : \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and $\underline{p}_i', \underline{p}_j', \underline{p}_k' \subset \mathcal{O}_{\mathbb{P}^1}(-1).$

The corresponding Gunning planes descend to

- $\underline{\Pi}_{i,j,k} = \{(\underline{E}, \underline{p}) ; \underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_i, \underline{p}_j, \underline{p}_k \subset \mathcal{O}_{\mathbb{P}^1}(-2) \subset \underline{E}\};$
- $\underline{\Pi}_{l,m,n} = \{(\underline{E}', \underline{p}') ; \underline{E}' = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2) \text{ and } \underline{p}_l', \underline{p}_m', \underline{p}_n' \subset \mathcal{O}_{\mathbb{P}^1}(-2) \subset \underline{E}'\}.$

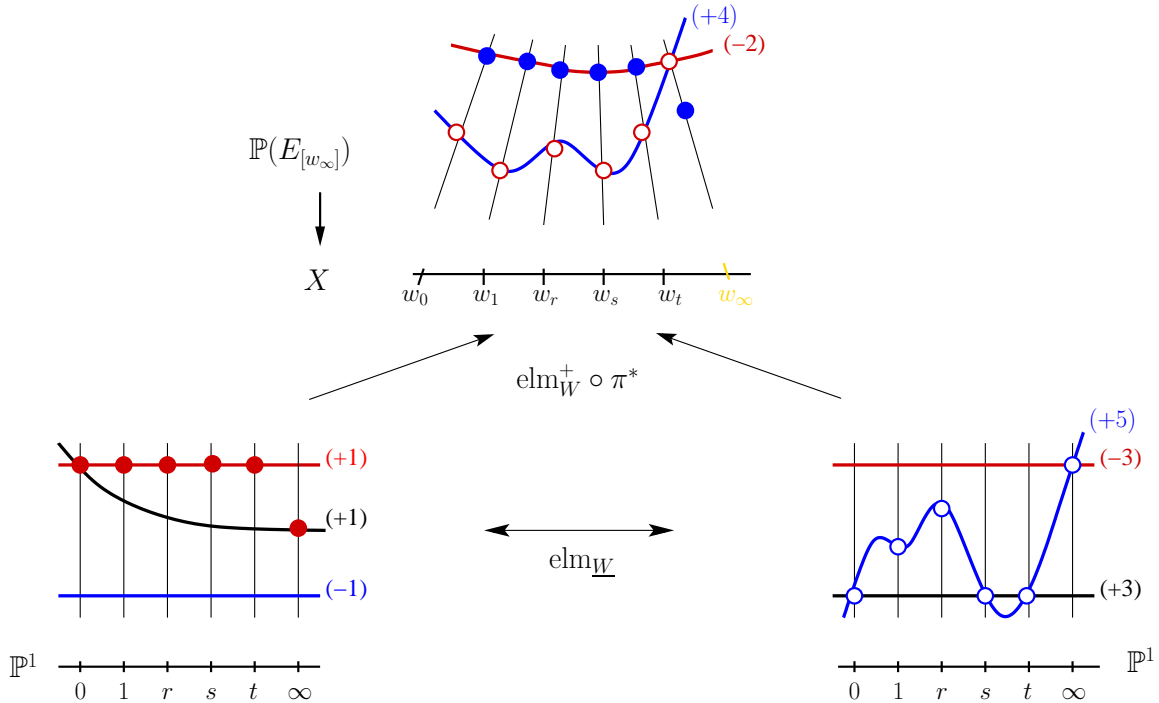
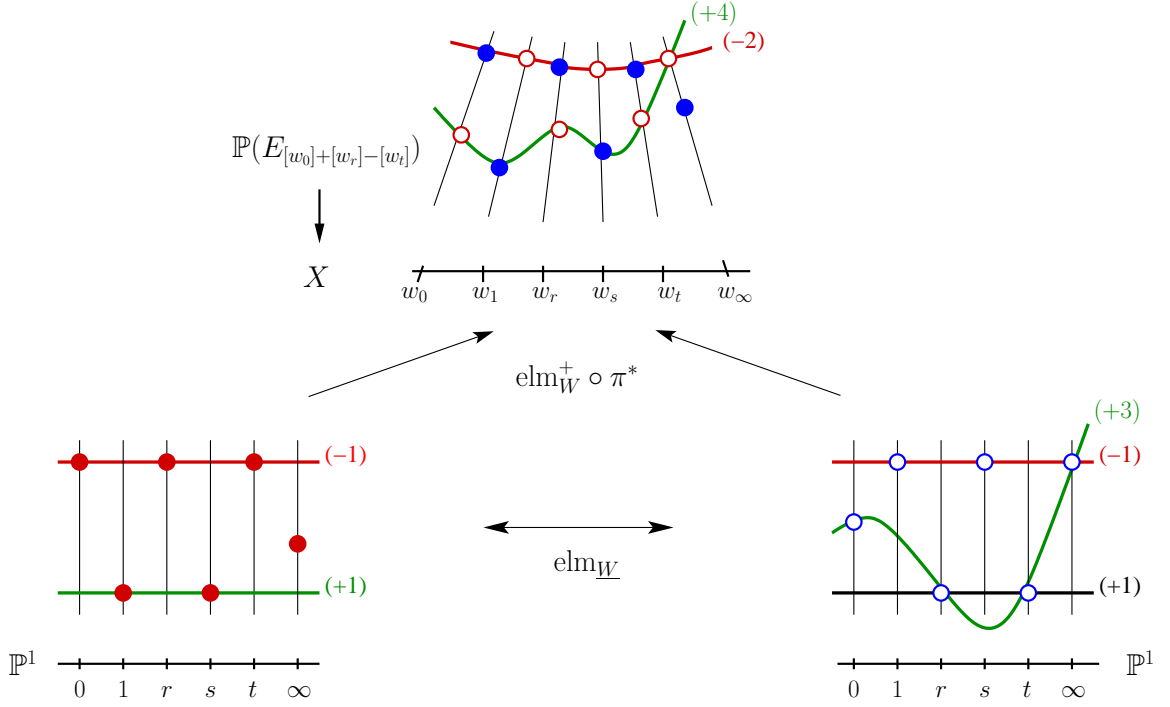


FIGURE 11. An odd Gunning bundle over X .

FIGURE 12. An even Gunning bundle over X .

5.3. Semi-stable bundles and projective charts. The coarse moduli space $\mathfrak{Bun}^{ind}(X/\iota)$ of rank 2 indecomposable parabolic bundles $(\underline{E}, \underline{p})$ over $\mathbb{P}^1 = X/\iota$ is studied in [2, 41]. From the previous section, $\mathfrak{Bun}(X/\iota) \setminus \mathfrak{Bun}^{ind}(X/\iota)$ only consists of 16 bundles, that correspond to the trivial bundle and its 15 twists on X (see Section 5.2.2). It turns out that a parabolic bundle $(\underline{E}, \underline{p})$ is indecomposable if, and only if, it is stable for a good choice of weights $\mu = (\mu_0, \mu_1, \mu_r, \mu_s, \mu_t, \mu_\infty) \in [0, 1]^6$ (see [41]). One can thus cover the moduli space $\mathfrak{Bun}^{ind}(X/\iota)$ by projective charts $\text{Bun}_\mu^{ss}(X/\iota)$ for a finite collection of weights, giving $\mathfrak{Bun}^{ind}(X/\iota)$ a structure of non separated scheme. In this context, two parabolic bundles $(\underline{E}, \underline{p})$ and $(\underline{E}', \underline{p}')$ are said to be *arbitrarily close* if there are families $(\underline{E}_t, \underline{p}_t)_{t \in \mathbb{A}^1}$ and $(\underline{E}'_t, \underline{p}'_t)_{t \in \mathbb{A}^1}$ such that $(\underline{E}_t, \underline{p}_t) \simeq (\underline{E}'_t, \underline{p}'_t)$ for each $t \neq 0$ but $(\underline{E}_0, \underline{p}_0) \simeq (\underline{E}, \underline{p})$ and $(\underline{E}'_0, \underline{p}'_0) \simeq (\underline{E}', \underline{p}')$. If two parabolic bundles over \mathbb{P}^1 are arbitrarily close then of course the corresponding vector bundles over X are arbitrarily close in the sense of Section 3.5. By the way, $\mathfrak{Bun}^{ind}(X/\iota)$ can be covered by charts isomorphic to $(\mathbb{P}^1)^3$ (see [2]) or \mathbb{P}^3 (see [41]). We provide a finite set of charts covering $\mathfrak{Bun}(X/\iota)$. We use two types of charts whose main representatives we now present:

5.3.1. The chart $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$. The first one (see [2] and [41] section 3.4) is given by weights of the form

$$\mu_0 = \mu_1 = \mu_\infty = \frac{1}{2} \quad \text{and} \quad \mu_r = \mu_s = \mu_t = 0$$

and is isomorphic to $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$. Precisely, μ -stable bundles $(\underline{E}, \underline{p})$ are given by $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ with $\underline{p}_0, \underline{p}_1, \underline{p}_\infty$ outside of $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$ and not all three

of them contained in the same $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$. Within the 2-parameter family of line subbundles isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-2)$ we can choose one containing at least \underline{p}_0 and \underline{p}_∞ say, and then choose meromorphic sections e_1 and e_2 of $\mathcal{O}_{\mathbb{P}^1}(-1)$ and $\mathcal{O}_{\mathbb{P}^1}(-2)$ (whose divisor is supported at $x = \infty$) such that the parabolic structure is normalized to

$$\underline{p}_i = \lambda_i e_1 + e_2 \quad \text{with} \quad (\lambda_0, \lambda_1, \lambda_\infty) = (0, 1, 0) \quad \text{and} \quad (\lambda_r, \lambda_s, \lambda_t) = (R, S, T) \in \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1.$$

To compare to the point of view of [2], note that

$$\mathcal{O}_{\mathbb{P}^1}(1) \otimes \text{elm}_\infty^+(\underline{E}, \underline{p}) = (\underline{E}'_0, \underline{p}')$$

is the trivial bundle $\underline{E}'_0 = \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}$ equipped with a parabolic structure having $\underline{p}'_0, \underline{p}'_1$ and \underline{p}'_∞ pairwise distinct (with respect to the trivialization of the bundle). From this chart, we can compute the two-fold cover $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}(X)$.

Proposition 5.2. *The classifying map $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_{\text{NR}}^3$ is explicitly given by $(R, S, T) \mapsto (v_0 : v_1 : v_2 : v_3)$ where*

$$v_0 = s^2 t^2 (r^2 - 1)(s - t)R - r^2 t^2 (s^2 - 1)(r - t)S + s^2 r^2 (t^2 - 1)(r - s)T + t^2 (t - 1)(r^2 - s^2)RS - s^2 (s - 1)(r^2 - t^2)RT + r^2 (r - 1)(s^2 - t^2)ST$$

$$v_1 = rst [((r - 1)(s - t)R - (s - 1)(r - t)S + (t - 1)(r - s)T + (t - 1)(r - s)RS - (s - 1)(r - t)RT + (r - 1)(s - t)ST)]$$

$$v_2 = -st(r^2 - 1)(s - t)R + rt(s^2 - 1)(r - t)S - rs(t^2 - 1)(r - s)T - t(t - 1)(r^2 - s^2)RS + s(s - 1)(r^2 - t^2)RT - r(r - 1)(s^2 - t^2)ST$$

$$v_3 = st(r - 1)(s - t)R - rt(s - 1)(r - t)S + sr(t - 1)(r - s)T + t(t - 1)(r - s)RS - s(s - 1)(r - t)RT + r(r - 1)(s - t)ST$$

This map is generically $(2 : 1)$ with indeterminacy points

$$(R, S, T) = (0, 0, 0), \quad (1, 1, 1), \quad (\infty, \infty, \infty) \quad \text{and} \quad (r, s, t).$$

The Galois involution $(R, S, T) \mapsto (\tilde{R}, \tilde{S}, \tilde{T})$ of this covering map is given by

$$\begin{aligned} \tilde{R} &= \lambda(R, S, T) \cdot \frac{(s-t)+(t-1)S-(s-1)T}{-t(s-1)S+s(t-1)T+(s-t)ST} \\ \tilde{S} &= \lambda(R, S, T) \cdot \frac{(r-t)+(t-1)R-(r-1)T}{-t(r-1)R+r(t-1)T+(r-t)RT} \\ \tilde{T} &= \lambda(R, S, T) \cdot \frac{(r-s)+(s-1)R-(r-1)S}{-s(r-1)R+r(s-1)S+(r-s)RS} \end{aligned}$$

$$\text{where} \quad \lambda(R, S, T) = \frac{t(r-s)RS-s(r-t)RT+r(s-t)ST}{(s-t)R-(r-t)S+(r-s)T}.$$

The ramification locus is over the Kummer surface; its lift on $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ is given by the equation

$$\begin{aligned} &((s - t)R + (t - r)S + (r - s)T)RST \\ &+ t((r - 1)S - (s - 1)R)RS + r((s - 1)T - (t - 1)S)ST + s((t - 1)R - (r - 1)T)RT \\ &- t(r - s)RS - r(s - t)ST - s(t - r)RT = 0. \end{aligned}$$

Proof. For computations, we work with the parabolic bundle

$$(\underline{E}'_0, \underline{p}') := \mathcal{O}_{\mathbb{P}^1}(1) \otimes \text{elm}_\infty^+(\underline{E}, \underline{p})$$

where $\underline{E}'_0 = \mathcal{O}_{\mathbb{P}^1} \otimes \mathcal{O}_{\mathbb{P}^1}$ is the trivial bundle, generated by sections e'_1 and e'_2 , and \underline{p}' is the parabolic structure defined by

$$\underline{p}'_i = \lambda_i e'_1 + e'_2 \quad \text{with} \quad (\lambda_0, \lambda_1, \lambda_\infty) = (0, 1, \infty) \quad \text{and} \quad (\lambda_r, \lambda_s, \lambda_t) = (R, S, T) \in \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1.$$

Let now E be the vector bundle over X obtained by

$$E := \text{elm}_W^+ (\pi^* (\underline{E}, \mathbf{p})) = \text{elm}_W^+ \left(\pi^* \left(\mathcal{O}_{\mathbb{P}^1}(-1) \otimes \text{elm}_{[\infty]}^- (\underline{E}'_0, \mathbf{p}') \right) \right);$$

this can be rewritten as

$$E := \mathcal{O}_X(-3[w_\infty]) \otimes \text{elm}_W^+ (\pi^* (\underline{E}'_0, \mathbf{p}')) = \mathcal{O}_X(-3[w_\infty]) \otimes \text{elm}_W^+ (E_0, \pi^* \mathbf{p}')$$

where E_0 is the trivial vector bundle on X .

In order to calculate the classifying map, we need to make the Narasimhan-Ramanan divisor D_E explicit in our coordinates. We may assume that E is generic (*i.e.* stable), so that D_E precisely describes the 1-parameter family of degree -1 line bundles $L \subset E$. After applying $\mathcal{O}_X(-3[\infty]) \otimes \text{elm}_W^+$, we get the family of degree -4 subbundles $L' \subset E_0$ (the trivial bundle over X) containing all 6 parabolics \mathbf{p}' . Precisely, if $L = \mathcal{O}_X([w_\infty] - [P_1] - [P_2])$, then $L' = \mathcal{O}_X(-3[w_\infty]) \otimes L = \mathcal{O}_X(-2[w_\infty] - [P_1] - [P_2])$. In other words, the Narasimhan-Ramanan divisor $D_E \subset \text{Pic}^1(X)$ is directly given by the 1-parameter family of points $\{P_1, P_2\}$ such that there is a line subbundle $L = \mathcal{O}_X(-[P_1] - [P_2] - 2[\infty]) \hookrightarrow E_0$ coinciding with the parabolic structure over W . Let $\sigma = (\sigma_1, \sigma_2) : X \rightarrow \mathbf{C}^2$ be a meromorphic section of L with divisor $-[P_1] - [P_2] - 2[\infty]$ with $P_i = (x_i, y_i) \in X$, $i = 1, 2$:

$$\begin{pmatrix} \sigma_1 \\ \sigma_2 \end{pmatrix} = \begin{pmatrix} \alpha + \beta x + \gamma \left(\frac{y-y_1}{x-x_1} - \frac{y-y_2}{x-x_2} \right) \\ \delta + \varepsilon x + \varphi \left(\frac{y-y_1}{x-x_1} - \frac{y-y_2}{x-x_2} \right) \end{pmatrix}.$$

After normalizing $\alpha = 1$, there is a unique choice of $\beta, \gamma, \delta, \varepsilon, \varphi \in \mathbb{C}$ such that

$$\sigma(0, 0) \parallel \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \sigma(1, 0) \parallel \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \sigma(r, 0) \parallel \begin{pmatrix} R \\ 1 \end{pmatrix}, \quad \sigma(s, 0) \parallel \begin{pmatrix} S \\ 1 \end{pmatrix}, \quad \sigma(\infty, \infty) \parallel \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The condition $\sigma(t, 0) \parallel \begin{pmatrix} T \\ 1 \end{pmatrix}$ depends now only on the choice of $\{P_1, P_2\}$ and writes (after convenient reduction)

$$v_0 \cdot 1 + v_1 \cdot \text{Sum}(P_1, P_2) + v_2 \cdot \text{Prod}(P_1, P_2) + v_3 \cdot \text{Diag}(P_1, P_2) = 0$$

with v_i as given in the proposition.

One can easily deduce that a generic point $(v_0 : v_1 : v_2 : v_3) \in \mathcal{M}_{\text{NR}}$ has precisely two preimages in $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$:

$$\begin{aligned} R &= \frac{r(t-1)(v_0+rv_1-r(s+t+st)v_3)T}{t(r-1)(v_0+tv_1-t(r+s+rs)v_3)-(r-t)(v_0+v_1-\sigma_2v_3)T} \\ S &= \frac{s(t-1)(v_0+sv_1-s(r+t+rt)v_3)T}{t(s-1)(v_0+tv_1-t(r+s+rs)v_3)-(s-t)(v_0+v_1-\sigma_2v_3)T}, \end{aligned}$$

where T is any solution of $aT^2 + bT + ct^2 = 0$ with

$$\begin{aligned} a &= (v_1 + v_2t + v_3t^2)(v_0 + v_1 - \sigma_2v_3) \\ b &= -(1+t)(v_0v_2 + v_1^2 + tv_1v_3) - 2(v_0v_1 + tv_0v_3 + tv_1v_2) \\ &\quad + \sigma_2(tv_1 + v_2 + tv_3)v_3 + (r+s+rs)(v_1 + t^2v_2 + t^2v_3)v_3 \\ c &= (v_1 + v_2 + v_3)(v_0 + tv_1 - t(r+s+rs)v_3). \end{aligned}$$

The discriminant of this polynomial leads again to our equation of the Kummer surface in the coordinates $(v_0 : v_1 : v_2 : v_3)$ given in Section 3.6. We can easily calculate the Galois involution of the classifying map $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_{\text{NR}}^3$. Its fixed points provide the equation in coordinates (R, S, T) of the lift of the Kummer surface. \square

5.3.2. *The chart \mathbb{P}_b^3 .* The other chart (namely the main chart \mathbb{P}_b^3 of [41]) is defined by democratic weights

$$\frac{1}{6} < \mu_0 = \mu_1 = \mu_r = \mu_s = \mu_t = \mu_\infty < \frac{1}{4}$$

and corresponds to the moduli space of the indecomposable parabolic structures on $\underline{E} := \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ having no parabolic in the total space of $\mathcal{O}_{\mathbb{P}^1}(-1)$. Parabolic bundles belonging to this chart are exactly those given by extensions

$$0 \rightarrow (\mathcal{O}_{\mathbb{P}^1}(-1), \emptyset) \rightarrow (\underline{E}, \underline{p}) \rightarrow (\mathcal{O}_{\mathbb{P}^1}(-2), \underline{W}) \rightarrow 0$$

i.e. elements of $\mathbb{P}H^1(\mathbb{P}^1, \text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-2) \otimes \mathcal{O}_{\mathbb{P}^1}(\underline{W}), \mathcal{O}_{\mathbb{P}^1}(-1)))$, which by Serre duality, identifies to $\mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^\vee$. After lifting them on $X \rightarrow \mathbb{P}^1$, applying elementary transformations and forgetting the parabolic structure, we precisely get those extensions

$$0 \rightarrow \mathcal{O}(-K_X) \rightarrow E \rightarrow \mathcal{O}(K_X) \rightarrow 0$$

i.e. by those points of $\mathbb{P}_B^4 = \mathbb{P}H^0(X, \mathcal{O}_X(3K_X))^\vee$, that are ι -invariant. Thus, the projective chart \mathbb{P}_b^3 of [41] naturally identifies with \mathbb{P}_B^3 introduced by Bertram (see Section 4.2). From this point of view, we have natural projective coordinates $\mathbf{b} = (b_0 : b_1 : b_2 : b_3)$, dual to the coordinates of ι -invariant cubic forms $(a_0 + a_1x + a_2x^2 + a_3x^3) \left(\frac{dx}{y}\right)^{\otimes 3}$. After computation, we get

Proposition 5.3. *The natural birational map $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ is given by*

$$(b_0 : b_1 : b_2 : b_3) \mapsto \begin{cases} R &= r \frac{b_3 - (s+t+1)b_2 + (st+s+t)b_1 - stb_0}{b_3 - \sigma_1 b_2 + \sigma_2 b_1 - \sigma_3 b_0} \\ S &= s \frac{b_3 - (r+t+1)b_2 + (rt+r+t)b_1 - rtb_0}{b_3 - \sigma_1 b_2 + \sigma_2 b_1 - \sigma_3 b_0} \\ T &= t \frac{b_3 - (r+s+1)b_2 + (rs+r+s)b_1 - rsb_0}{b_3 - \sigma_1 b_2 + \sigma_2 b_1 - \sigma_3 b_0} \end{cases}$$

The inverse map is given by $(R, S, T) \mapsto (b_0 : b_1 : b_2 : b_3)$ with

$$\begin{cases} b_0 &= \frac{R-r}{r(r-1)(r-s)(r-t)} + \frac{S-s}{s(s-1)(s-r)(s-t)} + \frac{T-t}{t(t-1)(t-r)(t-s)} \\ b_1 &= \frac{R-r}{(r-1)(r-s)(r-t)} + \frac{S-s}{(s-1)(s-r)(s-t)} + \frac{T-t}{(t-1)(t-r)(t-s)} \\ b_2 &= \frac{r(R-r)}{(r-1)(r-s)(r-t)} + \frac{s(S-s)}{(s-1)(s-r)(s-t)} + \frac{t(T-t)}{(t-1)(t-r)(t-s)} \\ b_3 &= \frac{r^2 R}{(r-1)(r-s)(r-t)} + \frac{s^2 S}{(s-1)(s-r)(s-t)} + \frac{t^2 T}{(t-1)(t-r)(t-s)} - \frac{1}{(r-1)(s-1)(t-1)} \end{cases}$$

This will be proved in Section 7.2, using Higgs fields. The geometry of this birational map is explained in section 5.4.1 and summarized in Figure 13.

Combination of the Propositions 5.3 and 5.2 yields

Corollary 5.4. *The natural map $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_{NR}^3$ is given by*

$$(b_0 : b_1 : b_2 : b_3) \mapsto \begin{cases} v_0 &= b_2 b_3 - (1 + \sigma_1) b_2^2 + (\sigma_1 + \sigma_2) b_1 b_2 - (\sigma_2 + \sigma_3) b_0 b_2 + \sigma_3 b_0 b_1 \\ v_1 &= b_2^2 - b_1 b_3 \\ v_2 &= b_0 b_3 - b_1 b_2 \\ v_3 &= b_1^2 - b_0 b_2 \end{cases}$$

Moreover, the (dual) Weddle surface, i.e. the lift to \mathbb{P}_B^3 of the Kummer equation, writes

$$\begin{aligned} &(-b_0 b_2 b_3^2 + b_1^2 b_3^2 + b_1 b_2^2 b_3 - b_2^4) + (1 + \sigma_1)(b_0 b_2^2 b_3 - 2b_1^2 b_2 b_3 + b_1 b_2^3) + (\sigma_1 + \sigma_2)(-b_0 b_2^3 + b_1^3 b_3) \\ &+ (\sigma_2 + \sigma_3)(-b_0 b_1^2 b_3 + 2b_0 b_1 b_2^2 - b_1^3 b_2) + \sigma_3(b_0^2 b_1 b_3 - b_0^2 b_2^2 - b_0 b_1^2 b_2 + b_1^4) = 0. \end{aligned}$$

This Corollary has to be compared to Section 4.2. Indeed, the components of the map $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_{\text{NR}}^3$ exactly correspond to the restriction to \mathbb{P}_B^3 of the natural quadratic forms on \mathbb{P}_B^4 vanishing along the embedding

$$X \hookrightarrow \mathbb{P}_B^4 ; (x, y) \mapsto (b_0 : b_1 : b_2 : b_3 : b_4) = (1 : x : x^2 : x^3 : y).$$

Indeed, the first one is the restriction of

$$b_4^2 - (b_2b_3 - (1 + \sigma_1)b_2^2 + (\sigma_1 + \sigma_2)b_1b_2 - (\sigma_2 + \sigma_3)b_0b_2 + \sigma_3b_0b_1)$$

which vanishes along $X \hookrightarrow \mathbb{P}_B^4$ from

$$y^2 = x(x-1)(x-r)(x-s)(x-t) = x^5 - (1 + \sigma_1)x^4 + (\sigma_1 + \sigma_2)x^3 - (\sigma_2 + \sigma_3)x^2 + \sigma_3x.$$

The other 3 quadratic forms just come from the following equalities on X

$$b_0b_2 = b_1^2 = x^2, \quad b_0b_3 = b_1b_2 = x^3 \quad \text{and} \quad b_1b_3 = b_2^2 = x^4.$$

It is quite surprising that the most natural basis both appearing from Bertram point of view, and Narasimhan-Ramanan point of view, are so compatible. They provide the same system of coordinate on \mathbb{P}_{NR}^3 which is however not considered in the classical theory of Kummer surfaces (see [34, 26]).

In Section 5.4 we provide four other charts, also with democratic weights, by varying μ in $[0, 1]$. This has the advantage, with respect to an arbitrary choice of a covering of $\mathfrak{Bun}(X/\iota)$ by charts $\mathfrak{Bun}_\mu^{ss}(X/\iota)$, to make the geometry of (birational) transition maps between charts quite clear.

5.3.3. Special bundles in the chart \mathbb{P}_b^3 . Here is the list of those special parabolic bundles of Section 5.2 that are semi-stable for $\frac{1}{6} < \mu_0 = \mu_1 = \mu_r = \mu_s = \mu_t = \mu_\infty < \frac{1}{4}$ and how they occur as special points in the chart \mathbb{P}_b^3 .

Proposition 5.5. *The only special bundles occuring (as semi-stable parabolic bundles) in $\text{Bun}_\mu^{ss}(X/\iota) = \mathbb{P}_b^3$ are generic bundles of the following families*

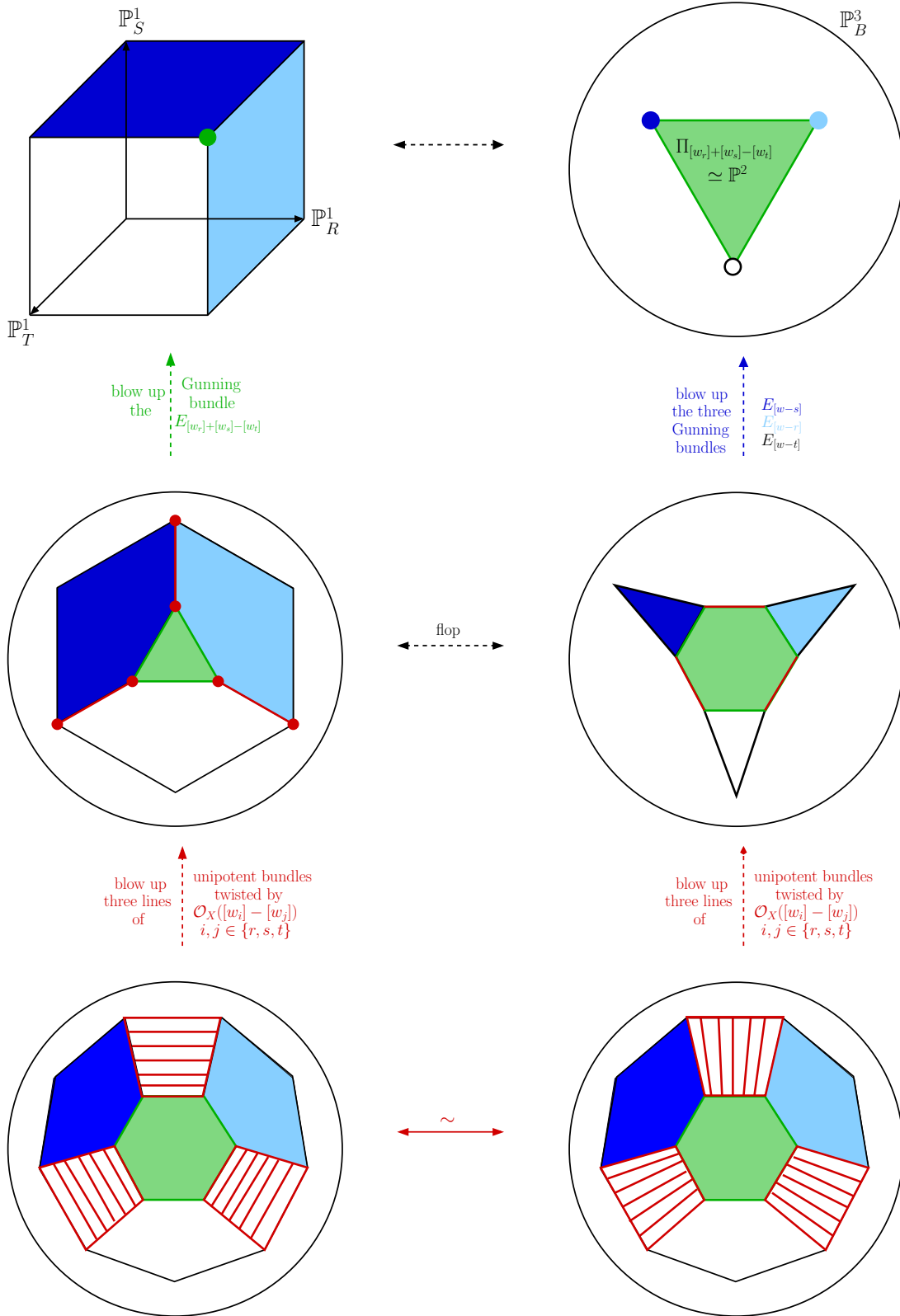
- **Unipotent bundles $\underline{\Delta}$:** *this 1-parameter family corresponds to the twisted cubic parametrized by*

$$X/\iota \rightarrow \mathbb{P}_b^3 ; x \mapsto (1 : x : x^2 : x^3).$$

- **Odd Gunning bundles Q_i :** *they are the 6 special points of the previous embedding $X/\iota \rightarrow \mathbb{P}_b^3$, namely Q_i is the image of the Weierstrass point w_i .*
- **twisted unipotent bundles $\underline{\Delta}_{i,j}$:** *lines of \mathbb{P}_b^3 passing through Q_i and Q_j .*
- **even Gunning planes $\underline{\Pi}_{i,j,k}$:** *planes of \mathbb{P}_b^3 passing through Q_i, Q_j and Q_k .*
- **odd Gunning planes $\underline{\Pi}'_i$:** *the quadric surface of \mathbb{P}_b^3 with a conic singular point at Q_i that contains the 5 lines $\Delta_{i,j}$ and the cubic Δ .*

Proof. It is easy to check which special parabolic bundles are semi-stable or not. For instance, the trivial bundle E_0 descends as the vector bundle $\underline{E}_0 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ equipped with the decomposable parabolic structure \underline{p} defined by the fibres of the line subbundle $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow \underline{E}_0$ (see Section 5.2.2); then $\mathcal{O}_{\mathbb{P}^1}$ is destabilizing.

Once this has been done, for each family occuring in \mathbb{P}_b^3 , we already know from Section 5.2 where they are sent on \mathbb{P}_{NR}^3 , we known the corresponding explicit equations from Section 3.6 and we can deduce equations on \mathbb{P}_b^3 by using explicit formula from Corollary 5.4. \square


 FIGURE 13. Geometry of the natural birational map $\mathbb{P}^3_B \dashrightarrow \mathbb{P}^1_R \times \mathbb{P}^1_S \times \mathbb{P}^1_T$.

Remark 5.6. *The proposition above is stated only for generic bundles of each type. Indeed, only an open set of the family $\underline{\delta}$ of unipotent bundles occurs in $\text{Bun}_{\mu}^{ss}(X/\iota) = \mathbb{P}_{\mathbf{b}}^3$, namely the complement of Weierstrass points (which are replaced by Gunning bundles Q_i). One can easily check that this is the only obstruction for the proposition to hold for all bundles of the respective families.*

The preimage of the Kummer surface $\text{Kum}(X)$ in the chart $\mathbb{P}_{\mathbf{b}}^3$ is nothing but the dual Weddle surface $\text{Wed}(X)$, another birational model of $\text{Kum}(X)$: it is also a quartic surface, but with only 6 nodes (see [34, 26]). Precisely, the 16 singular points of $\text{Kum}(X)$ are blown-up and replaced by the lines $\underline{\Delta}_{i,j}$; the 6 Gunning planes Π_i are contracted onto the points Q_i , giving rise to new conic points. In particular, all 16 quasi-unipotent families $\underline{\Delta}$ and $\underline{\Delta}_{i,j}$ are contained in Wed .

Actually, the map $\phi : \mathbb{P}_{\mathbf{b}}^3 \dashrightarrow \mathbb{P}_{\text{NR}}^3$ is defined by the linear system of quadrics passing through the 6 points Q_i ; indeed, for a general plane $\Pi \in \mathbb{P}_{\text{NR}}^3$, $\phi^*\Pi$ must intersect each contracted Π_i . We thus recover the quadric system in [17], §4.6. Those Π tangent to $\text{Kum}(X)$ have a singular lift $\underline{\Pi}$; when Π runs over the tangent planes of $\text{Kum}(X)$, the singular point of $\underline{\Pi}$ runs over the Weddle surface.

Remark 5.7. *The complement of the (dual) Weddle surface covers the open set of stable bundles in \mathbb{P}_{NR}^3*

$$\mathbb{P}_{\mathbf{b}}^3 \setminus \text{Wed}(X) \xrightarrow{\phi} \mathbb{P}_{\text{NR}}^3 \setminus \text{Kum}(X).$$

However, this is not a covering since over odd Gunning planes, only Π'_i occurs in $\mathbb{P}_{\mathbf{b}}^3$.

5.4. Moving weights and wall-crossing phenomena. For a generic weight μ , semi-stable bundles are automatically stable; in this case, the moduli space $\text{Bun}_{\mu}^{ss}(X/\iota)$ is projective, smooth and a geometric quotient. The special weights μ , for which some bundles are strictly semi-stable, form a finite collection of affine planes in the weight-space $[0, 1]^6 \ni \mu$ called *walls*. They cut-out $[0, 1]^6$ into finitely many *chambers*: the connected components of the complement of walls. Along walls, the moduli space is no more a geometric quotient, but a categorical quotient, identifying some semi-stable bundles together to get a (Hausdorff) projective manifold, which might be singular in this case; outside of the strictly semi-stable locus, $\text{Bun}_{\mu}^{ss}(X/\iota)$ is still smooth and a geometric quotient. The moduli space $\text{Bun}_{\mu}^{ss}(X/\iota)$ is constant in a given chamber; if not empty, it has the right dimension 3 and contains as an open set the geometric quotient of those bundles (E, \underline{p}) with $E = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and parabolics \underline{p} in general position:

- no parabolic in $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow E$,
- no 3 parabolics in the same $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow E$,
- no 5 parabolics in the same $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow E$.

Between (non empty!) moduli spaces in any two chambers, we get a natural birational map

$$\text{can} : \text{Bun}_{\mu}^{ss}(X/\iota) \xrightarrow{\sim} \text{Bun}_{\mu'}^{ss}(X/\iota)$$

arising from the identification of the generic bundles occurring in both of them. The indeterminacy locus comes from those special parabolic bundles that are stable for μ but not for μ' and *vice-versa*; this configuration occurs each time we cross a wall. The moduli space $\mathfrak{Bun}^{ind}(X/\iota)$ of indecomposable bundles can be covered by a finite collection of such moduli spaces, by choosing one μ in each non empty chamber; therefore, $\mathfrak{Bun}^{ind}(X/\iota)$ can be constructed by patching together these moduli spaces by means of

Possible configuration	(k, m, ℓ)	Walls
$\lambda = -\mu + \frac{5}{3}$	$(-5, 0, 0), (5, 3, 3)$	
$\lambda = -\mu + 1$	$(-3, 0, 0), (-1, 1, 1), (1, 2, 2), (3, 3, 3)$	
$\lambda = -\mu + \frac{1}{3}$	$(-1, 0, 0), (1, 3, 3)$	
$\lambda = -3\mu + 1$	$(-1, 0, 1), (1, 3, 2)$	③
$\lambda = -3\mu + 3$	$(-3, 0, 1), (3, 3, 2)$	
$\lambda = -\frac{1}{3}\mu + 1$	$(-3, 1, 0), (3, 2, 3)$	
$\lambda = -\frac{1}{3}\mu + \frac{1}{3}$	$(-1, 1, 0), (1, 2, 3)$	
$\lambda = 3\mu - 1$	$(-1, 0, 2), (1, 3, 1)$	②
$\lambda = \frac{1}{3}\mu + \frac{1}{3}$	$(-1, 2, 0), (1, 1, 3)$	
$\lambda = \mu + \frac{1}{3}$	$(-1, 3, 0), (1, 0, 3)$	
$\lambda = \mu - \frac{1}{3}$	$(-1, 0, 3), (1, 3, 0)$	①

Table 5: Possible wall-configurations for weights of the form $\mu = (\mu, \mu, \lambda, \lambda, \lambda, \mu)$.

canonical maps along the open set of common bundles. This gives $\mathfrak{Bun}^{ind}(X/\iota)$ a structure of smooth non separated scheme. However, in our case, we have also decomposable flat bundles that are not taken into account in this picture. For instance the preimage $(\underline{E}_0, \underline{p}^0) := \phi^{-1}(E_0)$ of the trivial bundle on X (see Section 5.2.2), being decomposable, can only arise as a singular point in semi-stable projective charts $\text{Bun}_{\mu}^{ss}(X/\iota)$. Indeed, if the bundle $\underline{E}_0 = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ equipped with the decomposable parabolic structure \underline{p}^0 defined by the fibres of $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow \underline{E}_0$ is semi-stable for some choice of weights μ , then all other parabolic structures \underline{p} on \underline{E}_0 with no parabolics in the total space of $\mathcal{O}_{\mathbb{P}^1} \subset \underline{E}_0$ are also semi-stable and arbitrarily close to \underline{p}^0 ; they are represented by the same point in the Hausdorff quotient $\text{Bun}_{\mu}^{ss}(X/\iota)$. One can check that this point is necessarily singular.

5.4.1. Wall-crossing between our two main charts. If we want to understand the geometry of the birational map $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ explicitly given in proposition 5.3 we have to consider a path in $\mathfrak{Bun}^{ind}(X/\iota)$ linking the corresponding chambers and the wall-crossing phenomena along this path. Since \mathbb{P}_B^3 corresponds to the weight $\mu = (\frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5}, \frac{1}{5})$ and $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ corresponds to the weight $\mu = (\frac{1}{2}, \frac{1}{2}, 0, 0, 0, \frac{1}{2})$, a possibility to do so consists in considering the walls between chambers of the form $\mu = (\mu, \mu, \lambda, \lambda, \lambda, \mu)$ with $\lambda, \mu \in [0, 1]$. A projective parabolic bundle belongs to such a wall if it possesses a section with self-intersection number $k \in 2\mathbb{Z} + 1$ containing m parabolics over $\{0, 1, \infty\}$ and ℓ parabolics over $\{r, s, t\}$ such that

$$0 = k + (3 - 2m)\mu + (3 - 2\ell)\lambda$$

for some $\lambda, \mu \in [0, 1]$. Table 5.4.1 lists all possible configurations. They are visualized in Figure 5.4.1

Following the pink line in Figure 5.4.1 means studying wall-crassing phenomena for moduli spaces $\text{Bun}_{\mu}^{ss}(X/\iota)$ with democratic weights $\mu = (\mu, \mu, \mu, \mu, \mu, \mu)$. As we see, walls occur for $\mu \in \{\frac{1}{6}, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \frac{5}{6}\}$. They will considered in Section 5.4.2.

First, we want to consider the crossing of the walls ①, ② and ③ in order to describe the birational map $\mathbb{P}_B^3 \dashrightarrow \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$. The configuration $(k, m, \ell) = (1, 3, 0)$ is not stable in $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$, but $(k, m, \ell) = (-1, 0, 3)$ is. It corresponds to the even Gunning bundle E_{ϑ} with $\vartheta = [w_r] + [w_s] - [w_t]$. The point in the moduli space $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$

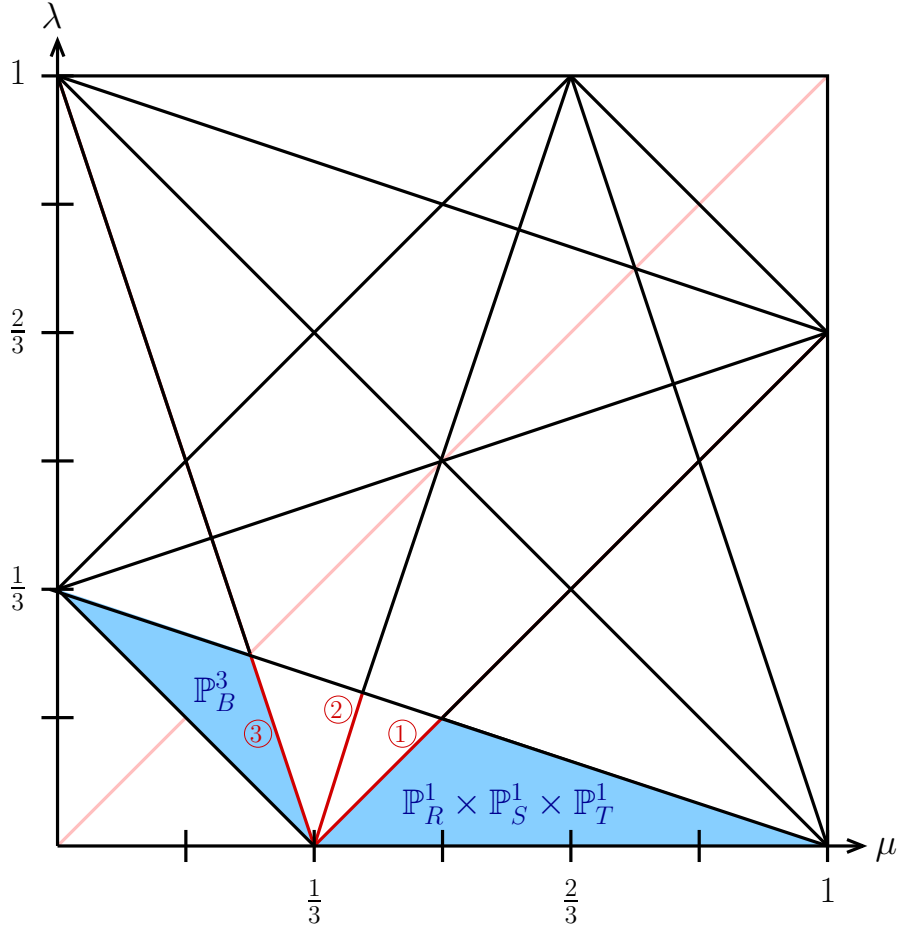


FIGURE 14. Chambers of moduli spaces for the weights $\mu = (\mu, \mu, \lambda, \lambda, \lambda, \mu)$.

corresponding to this bundle is blown up when crossing the wall ① and replaced by the corresponding Gunning plane: $(k, m, \ell) = (1, 3, 0)$. Passing on to wall ②, the three lines $(k, m, \ell) = (-1, 0, 2)$ in the moduli space corresponding to the unipotent bundles tensored by $\mathcal{O}_X([w_r] - [w_s])$, $\mathcal{O}_X([w_s] - [w_t])$ and $\mathcal{O}_X([w_s] - [w_t])$ respectively are no longer stable. Here a flop phenomenon occurs: these three lines are blown up and the resulting planes are contacted to three lines $(k, m, \ell) = (1, 3, 1)$ corresponding to the families of the same types of unipotent bundles. Passing on to wall ③, the three planes $(k, m, \ell) = (-1, 0, 1)$ corresponding to the odd Gunning planes with characteristic $\vartheta \in \{[w_r], [w_s], [w_t]\}$ are contracted and replaced by three points corresponding to the configurations $(k, m, \ell) = (1, 3, 2)$: the Gunning bundles with characteristic ϑ .

5.4.2. Democratic weights. Let us consider in this section the family of moduli spaces $\text{Bun}_{\mu}^{ss}(X/\iota)$ with weights $\mu = (\mu, \mu, \mu, \mu, \mu, \mu)$, for $\mu \in [0, 1]$. One can easily check which family of special bundle is semi-stable, depending on the choice of μ ; this is summarized in Table 6.

For $\mu \in [0, \frac{1}{6}]$. The moduli space $\text{Bun}_{\mu}^{ss}(X/\iota)$ is empty since $\mathcal{O}_{\mathbb{P}^1}(-1)$ is destabilizing the generic parabolic bundle (even if it carries no parabolic).

μ	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$	1
unipotent bundles (and twists)			Δ		Δ'		
			Δ_{ij}		Δ'_{ij}		
odd Gunning bundles and planes		Q_i	Π_i				
		Π'_i				Q'_i	
even Gunning planes		Π_{ijk}					

Table 6: Moving weights.

For $\mu = \frac{1}{6}$. The moduli space $\text{Bun}_{\mu}^{ss}(X/\iota)$ reduces to a single point. Indeed, it also contains the (non flat) decomposable bundle $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ with all parabolics \underline{p} lying in the total space of $\mathcal{O}_{\mathbb{P}^1}(-2)$. But the generic parabolic bundle is arbitrarily close to this decomposable bundle so that they have to be identified in the Hausdorff quotient $\text{Bun}_{\mu}^{ss}(X/\iota)$.

For $\mu \in]\frac{1}{6}, \frac{1}{4}[$. Here, we recover our chart $\mathbb{P}_{\mathbf{b}}^3 := \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota)$ with special families Δ , Δ_{ij} , Q_i , Π'_i and Π_{ijk} . The natural map $\phi : \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota) \dashrightarrow \mathbb{P}_{\text{NR}}^3$ has indeterminacy points at all 6 points Q_i .

For $\mu = \frac{1}{4}$. Now, odd Gunning planes Π_i become semi-stable, but arbitrarily close to the corresponding point Q_i , so that they are identified in the quotient $\text{Bun}_{\mu}^{ss}(X/\iota)$. Therefore, the moduli space is still the same $\mathbb{P}_{\mathbf{b}}^3$ but no more a geometric quotient.

For $\mu \in]\frac{1}{4}, \frac{1}{2}[$. Odd Gunning bundles Q_i are no more semi-stable and are replaced by the corresponding Gunning planes Π_i . The natural map

$$\text{can} : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \rightarrow \text{Bun}_{\frac{1}{6}, \frac{1}{4}}^{ss}(X/\iota)$$

is the blow-up of $\mathbb{P}_{\mathbf{b}}^3$ at all 6 points Q_i , and the exceptional divisors represent the corresponding planes Π_i . The natural map $\phi : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \rightarrow \mathbb{P}_{\text{NR}}^3$ is a morphism.

For $\mu = \frac{1}{2}$. the trivial bundle and its 15 twists become semi-stable (and just for this special value of μ). In particular, unipotent families are identified with these bundles in the moduli space, which has the effect to contract the strict transforms of lines Δ_{ij} and the rational curve Δ to 16 singular points of $\text{Bun}_{\mu}^{ss}(X/\iota)$. This moduli space is exactly the double cover of \mathbb{P}_{NR}^3 ramified along $\text{Kum}(X)$, therefore singular with conic points over each singular point of $\text{Kum}(X)$. The natural map

$$\text{can} : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \rightarrow \text{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$$

is a minimal resolution.

For $\mu \in]\frac{1}{2}, \frac{3}{4}[$. The families Δ and Δ_{ij} are no more semi-stable, and are replaced by the families Δ' and Δ'_{ij} . But mind that the canonical map

$$\text{can} : \text{Bun}_{\frac{1}{4}, \frac{1}{2}}^{ss}(X/\iota) \dashrightarrow \text{Bun}_{\frac{1}{2}, \frac{3}{4}}^{ss}(X/\iota)$$

is not biregular: there is a flop phenomenon around each of the 16 above rational curves. Precisely, after blowing-up the 16 curves, we exactly get the resolution $\widehat{\text{Bun}}_{\frac{1}{2}}^{ss}(X/\iota)$ of the previous moduli space by blowing-up the 16 conic points. Then, exceptional divisors are $\simeq \mathbb{P}^1 \times \mathbb{P}^1$ and we can contract them back to rational curves by using the other

ruling; this is the way the map can be constructed here. In particular, we get a second minimal resolution of $\text{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$.

For $\mu \in [\frac{3}{4}, \frac{5}{6}[$. Here, we finally contract the strict transforms of Π'_i to the points Q_i .

5.5. Galois and Geiser involutions. The Galois involution of the ramified cover $\phi : \mathfrak{Bun}(X/\iota) \xrightarrow{2:1} \mathfrak{Bun}(X)$

$$\Upsilon := \mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_{\underline{W}}^+ : \mathfrak{Bun}(X/\iota) \xrightarrow{\sim} \mathfrak{Bun}(X/\iota)$$

induces isomorphisms between moduli spaces

$$\Upsilon : \text{Bun}_{\mu}^{ss}(X/\iota) \xrightarrow{\sim} \text{Bun}_{\mu'}^{ss}(X/\iota)$$

where μ' is defined by $\mu'_i = \frac{1}{2} - \mu_i$ for all i . In particular, it underlines the symmetry of our special family of moduli spaces around $\mu = \frac{1}{2}$ (see Section 5.4): the Galois involution induces a biregular involution of $\text{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$, as well as isomorphisms

$$\text{Bun}_{\frac{1}{4}, \frac{1}{2}[}^{ss}(X/\iota) \xleftarrow{\sim} \text{Bun}_{\frac{1}{2}, \frac{3}{4}[}^{ss}(X/\iota) \quad \text{and} \quad \text{Bun}_{\frac{1}{6}, \frac{1}{4}[}^{ss}(X/\iota) \xleftarrow{\sim} \text{Bun}_{\frac{3}{4}, \frac{5}{6}[}^{ss}(X/\iota).$$

Considering now the composition

$$\text{Bun}_{\frac{1}{6}, \frac{1}{4}[}^{ss}(X/\iota) \xrightarrow{\text{can}} \text{Bun}_{\frac{3}{4}, \frac{5}{6}[}^{ss}(X/\iota) \xrightarrow{\Upsilon} \text{Bun}_{\frac{1}{6}, \frac{1}{4}[}^{ss}(X/\iota),$$

we get the (birational) Galois involution of the map $\phi : \mathbb{P}_{\mathbf{b}}^3 \dashrightarrow \mathbb{P}_{\text{NR}}^3$ described in Corollary 5.4. This is known as the Geiser involution (see [17], §4.6); it is a degree 7 birational map. The combination of all wall-crossing phenomena described in Section 5.4, when μ is varying from $\frac{1}{6}$ to $\frac{5}{6}$, provides a complete decomposition of this map (see Table 7):

- first blow-up 6 points (the Q_i along the embedding $X/\iota \xrightarrow{\sim} \underline{\Delta} \subset \mathbb{P}_{\mathbf{b}}^3$),
- flop 16 rational curves (the strict transforms of the twisted cubic $\underline{\Delta}$ and all lines $\underline{\Delta}_{ij}$),
- contract 6 planes (namely strict transforms of Π'_i onto Q_i),
- then compose by the unique isomorphism sending $Q'_i \rightarrow Q_i$.

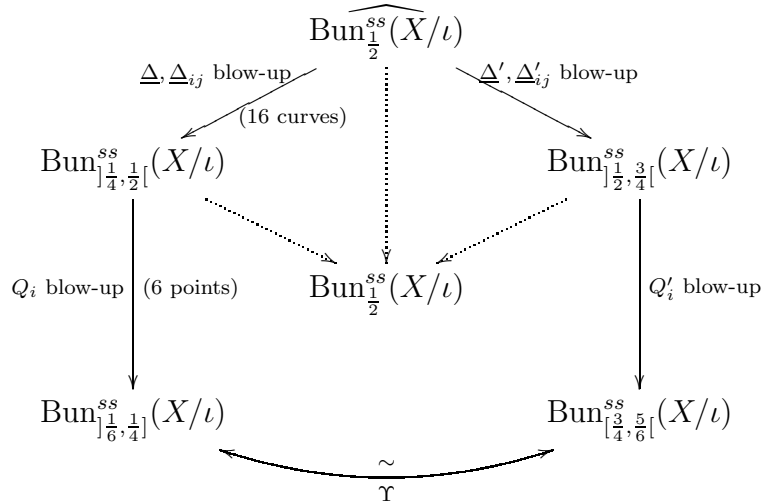


Table 7: Geometry of the Geiser involution.

Remark 5.8. *Even Gunning bundles Q_{ijk} are semi-stable if, and only if, $\mu = 1$. This is why they do not appear in our family of moduli spaces. However, for some other choices of weights μ , they appear as stable points, and therefore smooth points of some projective charts.*

5.6. Summary: the moduli stack $\mathfrak{Bun}(X)$. Recall from the introduction that we have defined $\mathfrak{Bun}(X/\iota)$ as the moduli space of parabolic rank 2 vector bundles with determinant $\mathcal{O}_X(-3)$ over \mathbb{P}^1 that can be endowed with a logarithmic connection (with poles over the Weierstrass points and prescribed residues). Denote by $\mathfrak{Bun}(X)$ the set of rank 2 vector bundles with trivial determinant over X that can be endowed with an tracefree holomorphic connection and $\mathfrak{Bun}^*(X)$ as the complement of the affine bundles in $\mathfrak{Bun}(X)$. The map $\phi : \mathfrak{Bun}(X/\iota) \rightarrow \mathfrak{Bun}^*(X)$ defined by the "hyperelliptic lift" $\text{elm}_W^+ \circ \pi^*$ (see Section 2) is surjective. Recall further that $\mathfrak{Bun}^{ind}(X/\iota)$ denotes the set of indecomposable parabolic bundles and that its image under ϕ is precisely the complement in $\mathfrak{Bun}^*(X)$ of the trivial bundle and its 15 twists (see Proposition 5.1 and Section 5.2.2).

As mentioned before, [41] provides methods to choose a finite set of smooth projective charts covering the moduli space $\mathfrak{Bun}^{ind}(X/\iota)$ of indecomposable parabolic bundles. In the present paper however, we chose to present charts with particular geometrical meaning and natural explicit coordinates. Similarly to the construction in 4.3, we can express explicitly the universal bundle in (affine parts of) each of these charts, giving $\mathfrak{Bun}(X)$ the structure of a moduli stack. Table 8 references the explicit maps between the charts given in this paper.

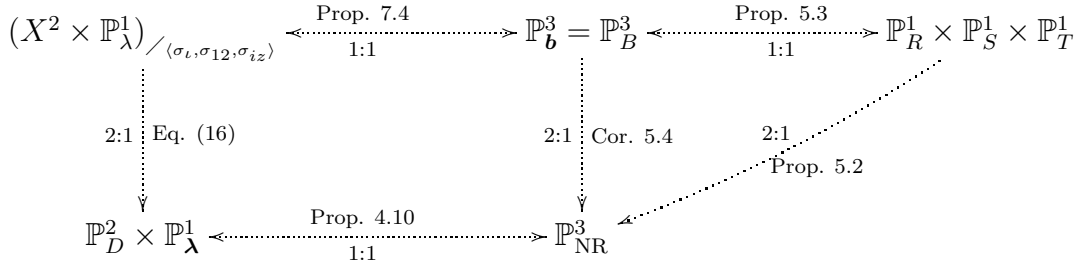


Table 8: Collection of explicit formulae.

As we can easily convince ourselves with the help of our dictionary in Section 5.2, Table 9 lists which elements of $\mathfrak{Bun}(X)$ occur in the image under ϕ of the respective charts. Here we use a checkmark sign (\checkmark) if every bundle of a given type can be found in the image of this chart and no checkmark sign if no bundle of the given type can be found in the image of this chart.

Consider the two democratic charts $\mathbb{P}^3_b = \text{Bun}_{\frac{1}{5}}^{ss}(X/\iota)$ and $\text{Bun}_{\frac{1}{3}}^{ss}(X/\iota)$. Their birational relation has been thoroughly described in Sections 5.4 and 5.5. According to Table 9, the union of the images under ϕ of these two charts covers the whole space $\mathfrak{Bun}^*(X)$ minus the ten even Gunning bundles and the decomposable bundles. Moreover, the Galois-involution $\Upsilon := \mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_W^+$ (see Section 5.5) sends $\mathbb{P}^3_b = \text{Bun}_{\frac{1}{5}}^{ss}(X/\iota)$ to $\Upsilon(\mathbb{P}^3_b) = \text{Bun}_{\frac{4}{5}}^{ss}(X/\iota)$ and $\text{Bun}_{\frac{1}{3}}^{ss}(X/\iota)$ to $\Upsilon(\mathbb{P}^3_b) = \text{Bun}_{\frac{2}{3}}^{ss}(X/\iota)$. Hence the two charts

$\mathbb{P}_b^3, \text{Bun}_{\frac{1}{3}}^{ss}(X/\iota)$ and their images under Υ are sufficient to cover $\mathfrak{Bun}^{ind}(X/\iota)$ minus the (pre-images of) even Gunning bundles.

Similar to the construction of our chart $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$, we can construct charts $\mathbb{P}_i^1 \times \mathbb{P}_j^1 \times \mathbb{P}_k^1$ for any choice of three distinct elements $i, j, k \in \{0, 1, r, s, t, \infty\}$ by setting $\mu_i = \mu_j = \mu_k = 0$ and all the other weights equal to $\frac{1}{2}$. The geometry and explicit formulae of the transition maps between these charts are obvious. Moreover, the explicit formulae in Proposition 5.2 can easily be generalized to any of these charts. According to Table 9, ten of these charts (one for each partition $\{i, j, k\} \cup \{k, l, m\} = \{0, 1, r, s, t, \infty\}$) are sufficient if we want their image to cover the whole space $\mathfrak{Bun}^*(X)$ minus the decomposable bundles. We can of course cover the whole space $\mathfrak{Bun}^*(X)$ by adding the singular chart $\text{Bun}_{\frac{1}{2}}^{ss}(X/\iota)$ to the 10 charts of type $\mathbb{P}_i^1 \times \mathbb{P}_j^1 \times \mathbb{P}_k^1$. Transition maps to this chart can be obtained from Proposition 5.3. Note however that even the union of all 20 charts of type $\mathbb{P}_i^1 \times \mathbb{P}_j^1 \times \mathbb{P}_k^1$ is however not sufficient to cover $\mathfrak{Bun}(X/\iota)$ minus the decomposable bundles (we don't have the images under the Galois involution of the Gunning bundles for example).

If we wish to cover $\mathfrak{Bun}(X)$ entirely (and not only $\mathfrak{Bun}^*(X)$), we have to add non-hyperelliptic charts, for example *via* the construction in Section 4.3. Indeed, there we have a parabolic structure over some divisor in $|2K_X|$ defined by four points in \mathbb{P}^1 . We can normalize three of them to 0, 1 and ∞ respectively, the fourth then is given by some $\tilde{\lambda}_j \in \mathbb{P}^1$. We deduce four natural charts

$$X^{(2)} \big/_{\{\iota, \iota\}} \times \mathbb{P}_{\tilde{\lambda}_j}^1 \xrightarrow{2:1} \mathfrak{Bun}(X) \quad \text{where} \quad j \in \{1, 2, 3, 4\}.$$

Here $\{\iota, \iota\}$ denotes the diagonal action $\{P, Q\} \mapsto \{\iota(P), \iota(Q)\}$ on the symmetric product $X^{(2)}$, which leaves the cross-ratio of the corresponding parabolics invariant.

6. THE MODULI STACK $\mathfrak{Higgs}(X)$ AND THE HITCHIN FIBRATION

A *Higgs bundle* on a Riemann surface X is a vector bundle $E \rightarrow X$ endowed with a *Higgs field*, i.e. an \mathcal{O}_X -linear morphism

$$\Theta : E \rightarrow E \otimes \Omega_X^1(D),$$

where D is an effective divisor. If D is reduced, then Θ is called *logarithmic* and for any $x \in D$, the residual morphism $\text{Res}_x(\Theta) \in \text{End}(E_x)$ is well-defined. As usual, we will only consider the case where E is a rank 2 vector bundle with trivial determinant bundle and Θ is trace-free. By definition, a holomorphic ($D = \emptyset$) and trace-free Higgs-field on E is an element of $H^0(X, \mathfrak{sl}(E) \otimes \Omega_X^1)$, which, by Serre duality, is isomorphic to $H^1(X, \mathfrak{sl}(E))^\vee$. On the other hand, stable bundles are *simple*: they possess no non-scalar automorphism. For such bundles E , the vector space $H^1(X, \mathfrak{sl}(E))$ is precisely the tangent space in E of our moduli space $\mathfrak{Bun}(X)$ of flat vector bundles over X . Therefore, *in restriction to the open set of stable bundles* the moduli space $\mathfrak{Higgs}(X)$ of Higgs bundles identifies in a natural way to

$$\mathfrak{Higgs}(X) := T^*\mathfrak{Bun}(X).$$

Just as naturally, we can define

$$\mathfrak{Higgs}(X/\iota) := T^*\mathfrak{Bun}(X/\iota),$$

but we need to clarify its meaning. Let $(\underline{E}, \underline{p})$ be a parabolic bundle in $\mathfrak{Bun}(X/\iota)$. Then $T_{(\underline{E}, \underline{p})}^*\mathfrak{Bun}(X/\iota) = H^0(X, \mathfrak{sl}(\underline{E}, \underline{p}) \otimes \Omega_{\mathbb{P}^1}^1)$, where $\mathfrak{sl}(\underline{E}, \underline{p})$ denotes the space of trace-free endomorphisms of \underline{E} leaving \underline{p} invariant. Now consider the image of the natural

bundle type		\mathbb{P}_b^3	$\text{Bun}_{\frac{1}{3}}^{ss}(X/_\iota)$	$\text{Bun}_{\frac{1}{2}}^{ss}(X/_\iota)$	$\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$	$\bigcup_{j=1}^4 X^{(2)} /_{\{\iota, \iota\}} \times \mathbb{P}_{\tilde{\lambda}_j}^1$
stable	off the odd Gunning planes	✓	✓	✓	✓	✓
	on the odd Gunning planes	✓	✓	✓	✓	
unipotent	generic	✓	✓	✓	✓	✓
	special, corresponding to $[w_r], [w_s]$ or $[w_t]$		✓	✓	✓	✓
	special, corresponding to $[w_0], [w_1]$ or $[w_\infty]$		✓	✓		✓
	twisted	✓	✓	✓	✓	✓
affine						✓
semi-stable decomposable $L_0 \oplus L_0^{-1}$	$L_0 = \mathcal{O}_X([P] - [Q])$ with $P, Q \in X \setminus W$ and $P \neq Q$	✓	✓	✓	✓	
	$L_0 = \mathcal{O}_X([P] - [w_i])$ with $P \in X \setminus W$ and $i \in \{0, 1, \infty\}$		✓	✓		
	$L_0 = \mathcal{O}_X([P] - [w_i])$ with $P \in X \setminus W$ and $i \in \{r, s, t\}$		✓	✓	✓	
	$L_0^{\otimes 2} = \mathcal{O}_X$			✓		
Gunning bundle	even, $\vartheta = [w_r] + [w_s] - [w_t]$ $\simeq [w_0] + [w_1] - [w_\infty]$				✓	
	even, other ϑ					
	odd, $\vartheta \in \{[w_1], [w_0], [w_\infty]\}$	✓			✓	✓
	odd, other ϑ	✓				✓

Table 9: Types of bundles that can be found in the image under ϕ of the charts \mathbb{P}_b^3 , $\text{Bun}_{\frac{1}{2}}^{ss}(X/_\iota)$, $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ and $X^{(2)} /_{\{\iota, \iota\}} \times \mathbb{P}_{\tilde{\lambda}_j}^1$ respectively.

embedding

$$H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p}) \otimes \Omega_{\mathbb{P}^1}^1) \hookrightarrow H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})).$$

Via the (meromorphic) gauge transformation

$$\mathcal{O}_{\mathbb{P}^1}(-3) \otimes \text{elm}_W^+ \in H^0(\mathbb{P}^1, \text{SL}(\underline{E} \otimes \mathcal{O}_{\mathbb{P}^1}(\underline{W}), \underline{p}))$$

it corresponds precisely to those logarithmic Higgs fields $\underline{\Theta}$ in $H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))$ that have *apparent singularities* in \underline{p} over \underline{W} : the residual matrices are congruent to $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and \underline{p} corresponds to their eigenvectors. We shall denote this set of apparent logarithmic Higgs fields on \underline{E} by

$$H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}} \simeq H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p}) \otimes \Omega_{\mathbb{P}^1}^1).$$

On the other hand, if we see $\mathfrak{Bun}(X/\iota)$ as a space of bundles E over X with a lift h of the hyperelliptic involution, then the space of h -invariant Higgs fields on E also naturally identifies to the cotangent space $T_{(E,h)}^* \mathfrak{Bun}(X/\iota)$. Indeed, let $(\underline{E}, \underline{p})$ be a parabolic bundle in $\mathfrak{Bun}(X/\iota)$ and consider the corresponding parabolic bundle $(E, \underline{p}) = \text{elm}_W^+(\pi^*(\underline{E}, \underline{p}))$ over X together with its unique isomorphism $h : E \simeq \iota^* E$ such that \underline{p} corresponds to the $+1$ -eigenspaces of h . Let $\underline{\Theta}$ be a logarithmic Higgs field in

$$T_{(E,p)}^* \mathfrak{Bun}(X/\iota) \simeq H^0(\mathbb{P}^1, \mathfrak{sl}(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}}.$$

The corresponding Higgs bundle $(E, \Theta) = \text{elm}_W^+(\pi^*(\underline{E}, \underline{p}))$ then is h -invariant and holomorphic by construction.

Similarly to the case of connections, we obtain

$$\pi_*(E, \Theta) = \bigoplus_{i=1}^2 (\underline{E}_i, \underline{\Theta}_i),$$

where $(\underline{E}_i, \underline{\Theta}_i)$ are apparent logarithmic Higgs bundles on \mathbb{P}^1 with $D = \underline{W}$.

6.1. A Poincaré family on the 2-fold cover $\mathfrak{Higgs}(X/\iota)$. Since we get a universal vector bundle on an open part of $\mathfrak{Bun}(X/\iota)$ for our moduli problem (for instance over \mathbb{P}_B^3 , see Section 4.2), we can expect to find a universal family of Higgs bundles (resp. connections) there, which we will now construct over an open subset of the projective chart $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$, namely when $(R, S, T) \in \mathbb{C}^3$ is finite.

For $(i, z_i) = (r, R), (s, S), (t, T)$, define the Higgs field Θ_i given on a trivial chart $(\mathbb{P}^1 \setminus \{\infty\}) \times \mathbb{C}^2$ of $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ by

$$\Theta_i := \frac{dx}{x} \begin{pmatrix} 0 & 0 \\ 1 - z_i & 0 \end{pmatrix} + \frac{dx}{x-1} \begin{pmatrix} z_i & -z_i \\ z_i & -z_i \end{pmatrix} + \frac{dx}{x-i} \begin{pmatrix} -z_i & z_i^2 \\ -1 & z_i \end{pmatrix}$$

These parabolic Higgs fields are independent over \mathbb{C} (they do not share the same poles) and any other Higgs field Θ on \underline{E} respecting the parabolic structure \underline{p} given by (R, S, T) is a linear combination of these Θ_i :

$$\Theta = c_r \Theta_r + c_s \Theta_s + c_t \Theta_t \quad \text{for unique } c_r, c_s, c_t \in \mathbb{C}.$$

These generators are chosen such that the coefficient $(2, 1)$ of Θ_i vanishes at $x = j$ and k where $\{i, j, k\} = \{r, s, t\}$. They are also very natural on our chart $\text{Bun}_\mu^{ss}(X/\iota) = \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ with $\mu \in]\frac{1}{6}, \frac{1}{4}[$. Indeed, for our choice of chart and generators, we precisely get:

Proposition 6.1. *The differential 1-form dz_i on the affine chart $(R, S, T) \in \mathbb{C}^3 \subset \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ identifies under Serre duality with the Higgs bundle $\Theta_i \in H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}}$ for $(i, z_i) = (r, R), (s, S), (t, T)$.*

Proof. In an intrinsic way, the tangent space of the moduli space of parabolic bundles at a point $(\underline{E}, \underline{p})$ is given by $H^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p}))$ where $\mathfrak{sl}(\underline{E}, \underline{p})$ is the sheaf of trace-free endomorphisms of E over \mathbb{P}_x^1 that preserve the parabolic structure. For instance the vector field $\frac{\partial}{\partial R} \in T_{(R,S,T)} \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ can be represented by the two charts $U_0 = \mathbb{P}_x^1 \setminus \{r\}$ and U_1 an analytic disc surrounding $x = r$ together with the cocycle

$$\phi_{0,1} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

on the punctured disc $U_{0,1} = U_0 \cap U_1$. Indeed, if we glue the restrictions $(\underline{E}, \underline{p})|_{U_0}$ and $(\underline{E}, \underline{p})|_{U_1}$ by the map

$$\exp(\zeta \phi_{0,1}) = \begin{pmatrix} 1 & 0 \\ \zeta & 1 \end{pmatrix} : ((\underline{E}, \underline{p})|_{U_1})|_{U_{0,1}} \rightarrow (\underline{E}, \underline{p})|_{U_0},$$

we get the new parabolic bundle defined by $\underline{p} = (0, 1, R + \zeta, S, T, \infty)$, *i.e.* the point defined by the time- ζ map generated by the vector field $\frac{\partial}{\partial R}$. Let us now compute the perfect pairing

$$\langle \cdot, \cdot \rangle : H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}} \times H^1(\mathbb{P}^1, \mathfrak{sl}(\underline{E}, \underline{p})) \rightarrow H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) \simeq \mathbb{C};$$

defining Serre duality in our coordinates. Given a Higgs field $\Theta \in H^0(\mathbb{P}^1, \mathfrak{sl}_2(\underline{E}) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\text{app}\underline{p}}$, the image in $H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1)$ is given by the cocycle

$$\langle \Theta, \phi_{0,1} \rangle = \text{trace}(\Theta \cdot \phi_{0,1})$$

on $U_{0,1}$, that is the $(1, 2)$ -coefficient of Θ restricted to $U_{0,1}$ (note that Θ is holomorphic there). We fix an isomorphism $H^1(\Omega_{\mathbb{P}^1}^1) \rightarrow \mathbb{C}$ as follows. Given a cocycle $(U_{0,1}, \omega_{0,1}) \in H^1(\Omega_{\mathbb{P}^1}^1)$, one can easily write $\omega_{0,1} = \alpha_0 - \alpha_1$ for meromorphic 1-forms α_i on U_i . Then $\omega_{0,1}$ is trivial in $H^1(\Omega_{\mathbb{P}^1}^1)$ if, and only if, $\omega_{0,1} = \omega_0 - \omega_1$ for holomorphic 1-forms ω_i on U_i , or, equivalently, if the principal part defined by $(\alpha_i)_i$ is that of a global meromorphic 1-form $(\alpha_i - \omega_i)_i$. Since the obstruction is given precisely by the Residue Theorem, we are led to define

$$\text{Res} : H^1(\mathbb{P}^1, \Omega_{\mathbb{P}^1}^1) \rightarrow \mathbb{C}$$

as the map which to a principal part $(\alpha_i)_i$ representing the cocycle, associates the sum of residues. For instance,

$$\omega_{0,1} := \langle \Theta_r, \phi_{0,1} \rangle = (1 - R) \frac{dx}{x} + R \frac{dx}{x-1} - \frac{dx}{x-r}$$

can be represented by the cocycle

$$\alpha_0 := 0 \quad \text{and} \quad \alpha_1 := -\omega_{0,1}$$

so that the principal part is just defined by $\frac{dx}{x-r}$ at $x = r$ and we get

$$\text{Res} \langle \Theta_r, \phi_{0,1} \rangle = 1$$

i.e. $\langle \Theta_r, \frac{\partial}{\partial R} \rangle = 1$. Similarly, we have

$$\left\langle \Theta_i, \frac{\partial}{\partial z_j} \right\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

□

Corollary 6.2. *The Liouville form on $T^*\text{Bun}_\mu^{ss}(X/\iota)$ defines a holomorphic symplectic 2-form on the moduli space of Higgs bundles defined in the chart $(R, S, T, c_r, c_s, c_t) \in \mathbb{C}^6$ by*

$$\omega = dR \wedge dc_r + dS \wedge dc_s + dT \wedge dc_t.$$

6.2. The Hitchin fibration. On the moduli space of Higgs bundles on X , the Hitchin fibration is defined by the map

$$\text{Hitch} : \mathfrak{Higgs}(X) \rightarrow H^0(X, 2K_X) ; (E, \Theta) \mapsto \det(\Theta).$$

Viewing $\mathfrak{Higgs}(X)$ as the total space of the cotangent bundle $T^*\text{Bun}(X)$ (over the open set of stable bundles), the Liouville form defines a symplectic structure on $\mathfrak{Higgs}(X)$. The above map defines a completely integrable system on this space: writing a quadratic differential as $(h_2x^2 + h_1x + h_0) \left(\frac{dx}{y}\right)^{\otimes 2}$, the 3 components of Hitch

$$h_0, h_1, h_2 : \mathfrak{Higgs}(X) \rightarrow \mathbb{C}$$

are holomorphic functions commuting to each other for the Poisson structure. Moreover, fibers of the map Hitch are (open sets of) 3-dimensional abelian varieties. One can also associate to (E, Θ) the spectral curve $\text{spec}(\Theta)$ which is the double-section of the projectivized bundle $\mathbb{P}E \rightarrow X$ defined by the eigendirections of Θ . This curve $\text{spec}(\Theta)$ is thus a two-fold ramified cover of X , ramifying at zeroes of the quadratic form $\text{Hitch}(E, \Theta)$; the spectral curve is thus constant along Hitchin fibers and its Jacobian is the compactification of the fiber (see for example [35]).

6.3. Explicit Hitchin Hamiltonians on $\mathfrak{Higgs}(X/\iota)$. Viewing a Higgs field as the difference of two connections, we have seen that Higgs bundles are invariant under involution and descend, likely as connections, as parabolic Higgs fields on $\mathbb{P}_x^1 = X/\iota$. The induced map

$$\mathfrak{Higgs}(X/\iota) \xrightarrow{2:1} \mathfrak{Higgs}(X)$$

allows us to compute the Hitchin fibration easily. Note that, applying an elementary transformation to some Higgs bundle (E, Θ) does not modify $\det(\Theta)$ since an elementary transformation is just a birational bundle transformation, acting by conjugacy on Θ . Therefore, to get Hitchin Hamiltonians on the chart (R, S, T, c_r, c_s, c_t) , we just have to compute

$$\det(c_r\Theta_r + c_s\Theta_s + c_t\Theta_t) = (h_2x^2 + h_1x + h_0) \frac{(dx)^{\otimes 2}}{x(x-1)(x-r)(x-s)(x-t)}.$$

A straightforward computation yields the explicit Hitchin Hamiltonians for $\mathfrak{Higgs}(X/\iota)$ given in Table 10.

It is easy to check that these functions indeed Poisson-commute: for any $f, g \in \{h_0, h_1, h_2\}$, we have

$$\sum_{i=r,s,t} \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} - \frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} = 0$$

in Darboux notation $(p_r, p_s, p_t, q_r, q_s, q_t) := (R, S, T, c_r, c_s, c_t)$.

In Proposition 5.3, we specified the birational map $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_B^3$, allowing us to express the Bertram coordinates $(b_0 : b_1 : b_2 : b_3)$ as functions of (R, S, T) . Setting

$$c_r dR + c_s dS + c_t dT = \lambda_1 d\frac{b_1}{b_0} + \lambda_2 d\frac{b_2}{b_0} + \lambda_3 d\frac{b_3}{b_0}$$

$$\begin{aligned}
h_0 &= (c_r(R-1) + c_s(S-1) + c_t(T-1)) (c_r st(R-r)R + c_s rt(S-s)S + c_t rs(T-t)T) \\
h_1 &= +c_r (c_r(s+t)(r+1) + c_s s(t+1) + c_t t(s+1)) R^2 - c_r^2 (t+s) R^3 \\
&\quad + c_s (c_s(r+t)(s+1) + c_r r(t+1) + c_t t(r+1)) S^2 - c_s^2 (t+r) S^3 \\
&\quad + c_t (c_t(r+s)(t+1) + c_r r(s+1) + c_s s(r+1)) T^2 - c_t^2 (r+s) T^3 \\
&\quad - c_r c_s (t(R-1+S-1) + r(S-s) + s(R-r)) RS \\
&\quad - c_r c_t (s(R-1+T-1) + r(T-t) + t(R-r)) RT \\
&\quad - c_s c_t (r(S-1+T-1) + s(T-t) + t(S-s)) ST \\
&\quad - (c_t t(r+s) + c_r r(s+t) + c_s s(r+t)) (c_r R + c_s S + c_t T) \\
h_2 &= (c_r(R-1)R + c_s(S-1)S + c_t(T-1)T) (c_r(R-r) + c_s(S-s) + c_t(T-t))
\end{aligned}$$

Table 10: Explicit Hitchin Hamiltonians for the chart $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ of $\mathfrak{Bun}(X/\iota)$

allows us to express the coefficients c_r, c_s, c_t as functions of the Bertram coordinates as well. The Hitchin map in Bertram coordinates then writes

$$(h_2 x^2 + h_1 x + h_0) \frac{(dx)^{\otimes 2}}{x(x-1)(x-r)(x-s)(x-t)}$$

where the Hitchin Hamiltonians h_0, h_1, h_2 are given explicitly in Table 11.

6.4. Explicit Hitchin Hamiltonians on $\mathfrak{Higgs}(X)$. We can now push-down formulae onto X to give the explicit Hitchin Hamiltonians on $\mathfrak{Higgs}(X) \simeq T^*\mathfrak{Bun}(X)$. In order to do this, we consider the natural rational map $\phi^* : T^*\mathbb{P}_{\text{NR}}^3 \dashrightarrow T^*\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ induced by the explicit map $\phi : \mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1 \dashrightarrow \mathbb{P}_{\text{NR}}^3$ of Proposition 5.2. Then, for a general section $\mu_0 d\left(\frac{v_0}{v_3}\right) + \mu_1 d\left(\frac{v_1}{v_3}\right) + \mu_2 d\left(\frac{v_2}{v_3}\right)$, the Hitchin Hamiltonians are given, after straightforward computation, by the explicit formula in Table 12.

In section 3.6, we introduced symmetric coordinates $(t_0 : t_1 : t_2 : t_3)$ of \mathcal{M}_{NR} given by

$$\begin{pmatrix} t_0 \\ t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} a & b & c & d \\ -b & a & d & -c \\ c & d & a & b \\ d & -c & -b & a \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 0 & -\sqrt{\sigma_3} \\ 0 & \sqrt{\sigma_4} & 0 & 0 \\ 0 & \sqrt{\sigma_3} & \sqrt{\sigma_3} & \sqrt{\sigma_3} \\ 0 & 0 & 0 & \sqrt{\sigma_3 \sigma_4} \end{pmatrix} \cdot \begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{pmatrix},$$

$$\text{where} \quad a = rst(r-s)\sqrt{\sigma_4} + t\sqrt{\rho_r \rho_s} - rt(r-1)\sqrt{\rho_s} - st\sqrt{\sigma_4 \rho_r}$$

$$b = -st(s-1)\sqrt{\rho_r} + rt\sqrt{\sigma_4 \rho_s}$$

$$c = t(r-s)\sqrt{\sigma_3 \sigma_4} - t(r-1)\sqrt{\sigma_3 \rho_s}$$

$$d = -t(r-1)(s-1)(r-s)\sqrt{\sigma_3} + t(s-1)\sqrt{\sigma_3 \rho_r}$$

$$\text{and} \quad \begin{aligned} \sqrt{\sigma_3}^2 &= rst, & \sqrt{\sigma_4}^2 &= (r-1)(s-1)(t-1), \\ \sqrt{\rho_r}^2 &= r(r-1)(r-s)(r-t), & \sqrt{\rho_s}^2 &= s(s-1)(s-r)(s-t). \end{aligned}$$

We obtain rational Hitchin Hamiltonians for the coordinates $(t_0 : t_1 : t_2 : t_3)$ given explicitly in Table 13 with respect to a general section $\eta_0 d\left(\frac{t_0}{t_3}\right) + \eta_1 d\left(\frac{t_1}{t_3}\right) + \eta_2 d\left(\frac{t_2}{t_3}\right) = \mu_0 d\left(\frac{v_0}{v_3}\right) + \mu_1 d\left(\frac{v_1}{v_3}\right) + \mu_2 d\left(\frac{v_2}{v_3}\right)$.

Table 11: Explicit Hitchin Hamiltonians for the coordinates $(b_0 : b_1 : b_2 : b_3)$ of \mathbb{P}_B^3 . Here b_{ij} denotes $b_i - b_j$.

$$\begin{aligned}
h_0 &= \frac{\lambda_1 b_1 + \lambda_2 b_2 + \lambda_3 b_3}{b_0^4} \cdot \begin{cases} -b_0 \sigma_3 \cdot [\lambda_1 b_0 b_{10} + \lambda_2 (b_0 b_{21} + b_1 b_{10}) + \lambda_3 (b_0 b_{32} + b_1 b_{21} + b_2 b_{10})] \\ +b_0 \sigma_2 \cdot [\lambda_1 b_1 b_{10} + \lambda_2 (b_2 b_{10} + b_1 b_{21}) + \lambda_3 (b_1 b_{32} + b_2 b_{21} + b_3 b_{10})] \\ -\sigma_1 \cdot [\lambda_1 b_1^2 b_{10} + \lambda_2 b_2 (b_1 b_{10} + b_0 b_{21}) + \lambda_3 (b_0 b_2 b_{32} + b_0 b_3 b_{21} + b_1 b_3 b_{10})] \\ +1 \cdot [\lambda_1 (b_1^2 b_{21} + b_2 (b_1^2 - b_0 b_2)) + \lambda_2 b_2 (b_1 b_{21} + b_2 b_{10}) + \lambda_3 b_3 (b_0 b_{32} + b_1 b_{21} + b_2 b_{10})] \end{cases} \\
h_1 &= \frac{1}{b_0^4} \cdot \begin{cases} b_0 \sigma_3 \cdot [\lambda_2^2 b_0 (b_{21}^2 - b_2 b_{20}) + \lambda_3^2 (-b_0 b_2 (2b_3 - b_2) - b_1 b_3 (b_1 - 2b_0)) - \lambda_1 \lambda_2 b_0 b_1 b_{10} - \lambda_1 \lambda_3 b_1^2 b_{10} \\ + \lambda_2 \lambda_3 (2b_0 b_1 b_2 + b_2 b_0 (b_0 - 2b_2) - b_0 b_3 (2b_1 - b_0) - b_2 b_{10}^2)] \\ +b_0 \sigma_2 \cdot [\lambda_2^2 b_2 (b_{10}^2 + b_0 b_{20}) + \lambda_3^2 b_3 (b_0 (b_3 - 2b_2) + b_1 (2b_2 - b_1)) + \lambda_1 \lambda_2 b_1^2 b_{10} \\ + \lambda_1 \lambda_3 (b_1^2 (2b_2 - b_1) - b_0 b_2^2) + \lambda_2 \lambda_3 (b_1^2 b_{32} + 2b_2^2 b_{10} + 2b_0 b_3 b_{21})] \\ +b_0 \sigma_1 \cdot [\lambda_2^2 b_2 (b_{21}^2 - b_2 b_{20}) + \lambda_3^2 b_3 (-b_2 (b_2 - 2b_1) - b_3 (2b_1 - b_0)) + \lambda_1 \lambda_2 (b_0 b_2^2 - b_1^2 (2b_2 - b_1)) \\ + \lambda_1 \lambda_3 (b_0 b_2 (2b_3 - b_2) - b_1 (b_{21}^2 + b_1 (2b_3 - b_1))) + \lambda_2 \lambda_3 (b_3 (b_1^2 - 2b_2 (2b_1 - b_0)) - b_2^2 (b_2 - 2b_1))] \\ +1 \cdot [\lambda_1^2 (b_1^2 (b_1^2 - 2b_0 b_2) + b_0^2 b_2^2) + \lambda_2^2 b_2^2 (b_{10}^2 + b_0 b_{20}) + \lambda_3^2 b_3 (-b_0 (b_2^2 + 2b_1 b_3) + b_3 (b_1^2 + 2b_0 b_2)) \\ + \lambda_1 \lambda_2 b_2 (2b_1^2 - b_0 b_2) b_{10} + \lambda_1 \lambda_3 (-b_0 b_1 b_2^2 + 2b_1^2 b_3 b_{10} - b_0^2 b_3 (b_3 - 2b_2)) \\ + \lambda_2 \lambda_3 (b_0 b_2^2 (3b_3 - b_2) + 2b_1 b_2 b_3 (b_1 - 2b_0))] \end{cases} \\
h_2 &= \frac{\lambda_3}{b_0^3} \cdot \begin{cases} \sigma_3 \cdot [-\lambda_1 b_0 b_1 b_{10} - b_0 \lambda_2 (b_1 b_{21} + b_2 b_{10}) - \lambda_3 b_0 (b_1 b_{32} + b_2 b_{21} + b_3 b_{10})] \\ +\sigma_2 \cdot [\lambda_1 b_1^2 b_{10} + \lambda_2 b_2 (b_1 b_{10} + b_0 b_{21}) + \lambda_3 (b_0 b_2 b_{32} + b_0 b_3 b_{21} + b_1 b_3 b_{10})] \\ -\sigma_1 \cdot [\lambda_1 (b_1^2 b_{21} + b_2 (b_1^2 - b_0 b_2)) + \lambda_2 b_2 (b_2 b_{10} + b_1 b_{21}) + \lambda_3 b_3 (b_0 b_{32} + b_1 b_{21} + b_2 b_{10})] \\ +1 \cdot [\lambda_1 ((b_1^2 - b_0 b_2) (2b_3 - b_2) + b_1 b_2 b_{21}) + \lambda_2 (b_2^2 (b_2 - 2b_1) + b_3 (2b_1 b_2 - b_0 b_3)) \\ + \lambda_3 (b_3^2 (2b_1 - b_0) + b_2 b_3 (b_2 - 2b_1))] \end{cases}
\end{aligned}$$

$$\begin{aligned}
h_0 &= \frac{1}{v_3^3} \cdot \left\{ \begin{aligned} &\mu_0^2 \cdot [v_0^3 - (2\sigma_{23}v_0 + \sigma_3v_1 - (\sigma_{12}\sigma_3 + \sigma_{23}^2)v_3)v_0v_3 + \sigma_3(\sigma_{23}v_1 + \sigma_3v_2 + (\sigma_3 - \sigma_{123}\sigma_2)v_3)v_3^2] \\ &+ v_1\mu_1^2 \cdot [v_0v_1 + \sigma_3v_2v_3] \\ &+ v_1\mu_2^2 \cdot [v_0v_3 + v_1v_2 + (1 + \sigma_1)v_1v_3 - \sigma_{23}v_3^2] \\ &+ \mu_0\mu_1 \cdot [2(v_0 - \sigma_{23}v_3)v_0v_1 + (v_0v_2 - (v_1 - \sigma_{12}v_3)v_1 - (\sigma_{23}v_2 + \sigma_3v_3)v_3)\sigma_3v_3] \\ &+ \mu_0\mu_2 \cdot [v_0^2v_2 + v_0v_1^2 + \sigma_{23}(\sigma_{23}v_2 + \sigma_3v_3)v_3^2 \\ &\quad - (\sigma_{12}v_1 + 2\sigma_{23}v_2 + \sigma_3v_3)v_0v_3 - (\sigma_{23}v_1 + 2\sigma_3v_2 + (\sigma_1\sigma_3 - \sigma_{123}\sigma_2 + 2\sigma_3)v_3)v_1v_3] \\ &+ v_1\mu_1\mu_2 \cdot [v_0v_2 + v_1^2 - (\sigma_{12}v_1 + \sigma_{23}v_2 - \sigma_3v_3)v_3] \end{aligned} \right. \\
h_1 &= \frac{1}{v_3^3} \cdot \left\{ \begin{aligned} &\mu_0^2 \cdot [2v_0^2v_1 - 2\sigma_{23}v_0v_1v_3 + \sigma_3v_0v_2v_3 - \sigma_3v_1^2v_3 + \sigma_{12}\sigma_3v_1v_3^2 - \sigma_3\sigma_{23}v_2v_3^2 - \sigma_3^2v_3^3] \\ &+ v_1\mu_1^2 \cdot [v_0v_2 + v_1^2 - \sigma_{12}v_1v_3 - \sigma_{23}v_2v_3 + \sigma_3v_3^2] \\ &+ \mu_2^2 \cdot [v_0v_2v_3 - v_1^2v_3 + 2v_1v_2^2 + 2(1 + \sigma_1)v_1v_2v_3 + \sigma_{12}v_1v_3^2 - \sigma_{23}v_2v_3^2 - \sigma_3v_3^2] \\ &+ \mu_0\mu_1 \cdot [v_0^2v_2 + (3v_0 - \sigma_{23}v_3)v_1^2 + ((\sigma_{123}\sigma_2 - (\sigma_1 + 2)\sigma_3)v_1 + \sigma_{23}^2v_2 + \sigma_{23}\sigma_3v_3)v_3^2 \\ &\quad - (\sigma_{12}v_1 + 2\sigma_{23}v_2 + \sigma_3v_3)v_0v_3] \\ &+ \mu_0\mu_2 \cdot [v_0(2v_0v_3 + 4v_1v_2 + 4(1 + \sigma_1)v_1v_3 + \sigma_{12}v_2v_3 - 2\sigma_{23}v_3^2) + \sigma_{12}(v_1 - \sigma_{12}v_3)v_1v_3 + 2\sigma_3v_2^2v_3 \\ &\quad - (\sigma_{123}\sigma_2 - (\sigma_1 + 2)\sigma_3)v_2v_3^2 + \sigma_{12}\sigma_3v_3^3] \\ &+ \mu_1\mu_2 \cdot [v_0v_2^2 + 3v_1^2v_2 + 2((1 + \sigma_1)v_1 - \sigma_{23}v_3)v_1v_3 - (\sigma_{12}v_1 + \sigma_{23}v_2 + \sigma_3v_3)v_2v_3] \end{aligned} \right. \\
h_2 &= \frac{1}{v_3^3} \cdot \left\{ \begin{aligned} &v_1\mu_0^2 \cdot [v_0v_1 + \sigma_3v_2v_3] \\ &+ v_1\mu_1^2 \cdot [v_0v_3 + v_1v_2 + (1 + \sigma_1)v_1v_3 - \sigma_{23}v_3^2] \\ &+ \mu_2^2 \cdot [v_0v_3^2 - v_1v_2v_3 + \sigma_{12}v_2v_3^2 + (1 + \sigma_1)v_2^2v_3 + v_2^3] \\ &+ v_1\mu_0\mu_1 \cdot [v_0v_2 + v_1^2 - \sigma_{12}v_1v_3 - \sigma_{23}v_2v_3 + \sigma_3v_3^2] \\ &+ \mu_0\mu_2 \cdot [-2v_0v_1v_3 + v_0v_2^2 + v_1^2v_2 - \sigma_{12}v_1v_2v_3 - \sigma_{23}v_2^2v_3 - \sigma_3v_2v_3^2] \\ &+ \mu_1\mu_2 \cdot [v_0v_2v_3 - v_1^2v_3 + 2v_1v_2^2 + 2(1 + \sigma_1)v_1v_2v_3 + \sigma_{12}v_1v_3^2 - \sigma_{23}v_2v_3^2 - \sigma_3v_3^3] \end{aligned} \right.
\end{aligned}$$

Table 12: Explicit Hitchin Hamiltonians for the coordinates $(v_0 : v_1 : v_2 : v_3)$ of \mathcal{M}_{NR} . Here we denote $\sigma_{ij} = \sigma_i + \sigma_j$, $ij = 12, 13, 23$, and $\sigma_{123} = \sigma_1 + \sigma_2 + \sigma_3$, where $\sigma_1 = r + s + t$, $\sigma_2 = rs + st + tr$ and $\sigma_3 = rst$ as usual.

$$\begin{aligned}
h_0 &= \frac{1}{4t_3^4} \cdot \begin{cases} rst \cdot [\eta_0(t_0^2 - t_3^2) + \eta_1(t_0t_1 + t_2t_3) + \eta_2(t_0t_2 + t_1t_3)]^2 \\ -st \cdot [\eta_0(t_0t_1 - t_2t_3) + \eta_1(t_1^2 + t_3^2) + \eta_2(t_0t_3 + t_1t_2)]^2 \\ +4rs \cdot (\eta_0t_0 + \eta_1t_1)^2 t_3^2 \\ -rt \cdot [\eta_0(t_0^2 + t_3^2) + \eta_1(t_0t_1 + t_2t_3) + \eta_2(t_0t_2 - t_1t_3)]^2 \end{cases} \\
h_1 &= \frac{1}{4t_3^4} \cdot \begin{cases} t \cdot (t_0^2 + t_1^2 + t_2^2 + t_3^2) [(\eta_0^2 + \eta_1^2 + \eta_2^2)t_3^2 + (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)^2] \\ +st \cdot (t_0^2 - t_1^2 + t_2^2 - t_3^2) [(\eta_0^2 - \eta_1^2 + \eta_2^2)t_3^2 - (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)^2] \\ +4r \cdot (t_0t_2 - t_1t_3) t_3 [\eta_0\eta_2t_3 + (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)\eta_1] \\ +4sr \cdot (t_0t_2 + t_1t_3) t_3 [\eta_0\eta_2t_3 - (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)\eta_1] \\ +4s \cdot (t_0t_3 + t_1t_2) t_3 [\eta_1\eta_2t_3 - (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)\eta_0] \\ +4rt \cdot (t_0t_1 + t_2t_3) t_3 [\eta_0\eta_1t_3 - (\eta_0t_0 + \eta_1t_1 + \eta_2t_2)\eta_2] \end{cases} \\
h_2 &= \frac{1}{4t_3^4} \cdot \begin{cases} s \cdot [\eta_0(t_0t_2 + t_1t_3) + \eta_1(t_0t_3 + t_1t_2) + \eta_2(t_2^2 - t_3^2)]^2 \\ -1 \cdot [\eta_0(t_0t_2 - t_1t_3) + \eta_1(t_0t_3 + t_1t_2) + \eta_2(t_2^2 + t_3^2)]^2 \\ -t \cdot [\eta_0(t_0t_1 + t_3t_3) - \eta_2(t_0t_3 - t_1t_2) + \eta_1(t_2^2 + t_3^2)]^2 \\ +4r \cdot (\eta_1t_1 + \eta_2t_2)^2 t_3^2 \end{cases}
\end{aligned}$$

Table 13: Explicit Hitchin Hamiltonians for the coordinates $(t_0 : t_1 : t_2 : t_3)$ of \mathcal{M}_{NR} .

6.5. Comparison to existing formulae. In [23], B. van Geemen and E. Previato conjectured a projective version of explicit Hitchin Hamiltonians (up to multiplication by functions from the base), which has been confirmed in [22]. These Hamiltonians are expressed in symmetric coordinates $(t_0 : t_1 : t_2 : t_3)$ of \mathcal{M}_{NR} and with respect to a genus 2 curve X given by

$$y^2 = \prod_{i=1}^6 (x - \lambda_i).$$

The coefficients A, B, C, D of the Kummer surface (13) can be made explicit, allowing us to uniquely identify

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6) = (0, t, 1, s, r, \infty)$$

with respect to our coordinates. Indeed,

value of	in the paper [23]	in equation (14)
A	$2 \frac{(2(\lambda_1\lambda_2 + \lambda_3\lambda_4) - (\lambda_1 + \lambda_2)(\lambda_3 + \lambda_4))}{(\lambda_3 - \lambda_4)(\lambda_1 - \lambda_2)}$	$-2 \frac{s(t-1) + (t-s)}{t(s-1)}$
B	$-2 \frac{2(\lambda_1\lambda_2 + \lambda_5\lambda_6) - (\lambda_1 + \lambda_2)(\lambda_5 + \lambda_6)}{(\lambda_5 - \lambda_6)(\lambda_1 - \lambda_2)}$	$-2 \frac{r + (r-t)}{t}$
C	$2 \frac{2(\lambda_3\lambda_4 + \lambda_5\lambda_6) - (\lambda_3 + \lambda_4)(\lambda_5 + \lambda_6)}{(\lambda_5 - \lambda_6)(\lambda_3 - \lambda_4)}$	$2 \frac{(r-1) + (r-s)}{s-1}$
D	$-4 \frac{(\lambda_1 + \lambda_2)(\lambda_5\lambda_6 - \lambda_3\lambda_4) + (\lambda_3 + \lambda_4)(\lambda_1\lambda_2 - \lambda_5\lambda_6) + (\lambda_5 + \lambda_6)(\lambda_3\lambda_4 - \lambda_1\lambda_2)}{(\lambda_5 - \lambda_6)(\lambda_3 - \lambda_4)(\lambda_1 - \lambda_2)}$	$-4 \frac{r(s-t) + (r-s)}{t(s-1)}$

Let us denote

$$h(x) := h_2 x^2 + h_1 x + h_0,$$

where h_i for $i \in \{0, 1, 2\}$ are the Hitchin Hamiltonians given with respect to the symmetric coordinates in the affine chart $\left(\frac{t_0}{t_3} : \frac{t_1}{t_3} : \frac{t_2}{t_3} : 1\right)$ as in Table 13. The Hamiltonians H_1, \dots, H_6 in [23] can then be expressed in terms of the Hitchin Hamiltonians as

$$\begin{aligned} H_1 &= \frac{4h(0)}{rst} & H_4 &= \frac{4h(s)}{s(s-1)(s-r)(s-t)} \\ H_2 &= -\frac{4h(t)}{t(t-1)(t-r)(t-s)} & H_5 &= \frac{4h(r)}{r(r-1)(r-s)(r-t)} \\ H_3 &= \frac{4h(1)}{(r-1)(s-1)(t-1)} & H_6 &= 0 \end{aligned}$$

7. THE MODULI STACK $\mathfrak{Con}(X)$

Note that if ∇_1 and ∇_2 are connections on the same vector bundle $E \rightarrow X$, then $(E, \nabla_1 - \nabla_2)$ is a Higgs bundle. Hence $\mathfrak{Con}(X)$ (resp. $\mathfrak{Con}(X/\iota)$) can be seen as an affine extension of $\mathfrak{Higgs}(X)$ (resp. $\mathfrak{Higgs}(X/\iota)$). One connection on the parabolic bundle $(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2), \underline{p})$ attached to a parameter (R, S, T) is given in the affine chart $(\mathbb{P}^1 \setminus \{\infty\}) \times \mathbb{C}^2$ with coordinates (x, Y) by

$$\begin{aligned} (17) \quad \nabla_0 &:= d + \begin{pmatrix} 0 & 0 \\ -1 & \frac{1}{2} \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} 1 & -\frac{1}{2} \\ 1 & -\frac{1}{2} \end{pmatrix} \frac{dx}{x-1} \\ &+ \frac{1}{2} \begin{pmatrix} 0 & R \\ 0 & 1 \end{pmatrix} \frac{dx}{x-r} + \frac{1}{2} \begin{pmatrix} 0 & S \\ 0 & 1 \end{pmatrix} \frac{dx}{x-s} + \frac{1}{2} \begin{pmatrix} 0 & T \\ 0 & 1 \end{pmatrix} \frac{dx}{x-t} \end{aligned}$$

and hence in the affine chart $(\mathbb{P}^1 \setminus \{0\}) \times \mathbb{C}^2$ with coordinates $(\tilde{x}, \tilde{Y}) = \left(\frac{1}{x}, \begin{pmatrix} x & 0 \\ 0 & x^2 \end{pmatrix} Y\right)$ its residual part at $\tilde{x} = 0$ is given by $d + \begin{pmatrix} 0 & 0 \\ -1 & \frac{1}{2} \end{pmatrix} \frac{d\tilde{x}}{\tilde{x}}$. Any other connection on this bundle writes uniquely as

$$\nabla = \nabla_0 + c_r \Theta_r + c_s \Theta_s + c_t \Theta_t,$$

where the Higgs bundles Θ_i are defined in Section 6.1. This provides a universal family of parabolic connections on a large open subset of the moduli space. Note that the residual part at infinity is given for such a connection by

$$d + \begin{pmatrix} 0 & 0 \\ -1 - c_r(R-r) - c_s(S-s) - c_t(T-t) & \frac{1}{2} \end{pmatrix} \frac{d\tilde{x}}{\tilde{x}}.$$

7.1. An explicit atlas. We can use the above construction to cover the moduli space $\mathfrak{Con}^*(X/\iota)$ by affine charts, in each of which we can explicitly describe the Poincaré family. Here $\mathfrak{Con}^*(X/\iota)$ denotes the space of all those parabolic connections $(\underline{E}, \underline{\nabla})$ such that $\Phi(\underline{E}, \underline{\nabla})$ is not a twist of the trivial connection on the trivial vector bundle over X .

The map Φ then induces on the set $\mathfrak{Con}^*(X)$ of irreducible or abelian but non trivial $\mathfrak{sl}_2\mathbb{C}$ -connections on X the structure of a moduli stack.

Fix exponents $\kappa_i \in \mathbb{C}$ for

$$i \in \{0, 1, r, s, t, \infty\}$$

and define $\rho \in \mathbb{C}$ by

$$\kappa_0 + \kappa_1 + \kappa_r + \kappa_s + \kappa_t + \kappa_\infty + 2\rho = 1.$$

The universal connection on $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ with eigenvalues

$$(18) \quad \begin{pmatrix} x=0 & x=1 & x=r & x=s & x=t & x=\infty \\ 0 & 0 & 0 & 0 & 0 & \rho \\ \kappa_0 & \kappa_1 & \kappa_r & \kappa_s & \kappa_t & \kappa_\infty + \rho \end{pmatrix}$$

can be written as follows:

$$(19) \quad \nabla = \nabla_0 + c_r \Theta_r + c_s \Theta_s + c_t \Theta_t$$

with

$$\nabla_0 = d + \begin{pmatrix} 0 & 0 \\ \rho & \kappa_0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} -\rho & \rho + \kappa_1 \\ -\rho & \rho + \kappa_1 \end{pmatrix} \frac{dx}{x-1} + \sum_{i \in \{r,s,t\}} \begin{pmatrix} 0 & z_i \kappa_i \\ 0 & \kappa_i \end{pmatrix} \frac{dx}{x-i}$$

and

$$\Theta_i = \begin{pmatrix} 0 & 0 \\ 1 - z_i & 0 \end{pmatrix} \frac{dx}{x} + \begin{pmatrix} z_i & -z_i \\ z_i & -z_i \end{pmatrix} \frac{dx}{x-1} + \begin{pmatrix} -z_i & z_i^2 \\ -1 & z_i \end{pmatrix} \frac{dx}{x-i}$$

Here we have normalized the parabolic data to

$$(20) \quad \begin{matrix} x=0 & x=1 & x=r & x=s & x=t & x=\infty \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} z_r \\ 1 \end{pmatrix} & \begin{pmatrix} z_s \\ 1 \end{pmatrix} & \begin{pmatrix} z_t \\ 1 \end{pmatrix} & \mathcal{O}_{\mathbb{P}^1}(-1) \end{matrix}$$

We note that eigendirections with respect to 0-eigenvalue are generated by

$$\begin{pmatrix} x=0 \\ -\frac{\kappa_0}{\rho + \sum_{i \in \{r,s,t\}} c_i(z_i-1)} \\ 1 \end{pmatrix} \quad \begin{pmatrix} x=1 \\ 1 + \frac{\kappa_1}{\rho + \sum_{i \in \{r,s,t\}} c_i z_i} \\ 1 \end{pmatrix} \quad \begin{pmatrix} x=r \\ z_r - \frac{\kappa_r}{c_r} \\ 1 \end{pmatrix} \quad \begin{pmatrix} x=s \\ z_s - \frac{\kappa_s}{c_s} \\ 1 \end{pmatrix} \quad \begin{pmatrix} x=t \\ z_t - \frac{\kappa_t}{c_t} \\ 1 \end{pmatrix}$$

This matrix connection can be thought of, via an elementary transformation at $x = \infty$, as a parabolic system (a parabolic connection on the trivial bundle) with shifted eigenvalues $(\rho, \rho + \kappa_\infty - 1)$ and parabolic now associated to ρ normalized to $e_1 = {}^t(1, 0)$. Similarly, after twisting by the (unique) rank 1 connection $(\mathcal{O}_{\mathbb{P}^1}(-1), \zeta)$ having a single pole at infinity, we get a universal family for those connections on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ with shifted eigenvalues $(\rho + 1, \rho + \kappa_\infty + 1)$ at $x = \infty$.

We obtain an atlas of charts of $\mathfrak{Con}^*(X/\iota)$, each possessing a universal connection, as follows (the birational transition maps between charts are straightforward to calculate and will not be carried out explicitly):

- **Canonical chart:** When $\kappa_i = \frac{1}{2}$ for all $i \in \underline{W}$ (and thus $\rho = -1$), we obtain our first affine chart

$$(z_r, z_s, z_t, c_r, c_s, c_t) \in \mathbb{C}^6 =: U_0$$

together with its universal family of connections (19).

- **Switch:** Choose $J \subset \{r, s, t\}$. Set $\kappa_j = -\frac{1}{2}$ for all $j \in J$ and $\kappa_i = \frac{1}{2}$ for all $i \in \underline{W} \setminus J$. Tensorize the corresponding universal family of connections (19) with the (unique) logarithmic rank 1 connection on $\eta : \mathcal{O}_{\mathbb{P}^1} \rightarrow \mathcal{O}_{\mathbb{P}^1} \otimes \Omega_{\mathbb{P}^1}^1(J + [\infty])$ having eigenvalues $+\frac{1}{2}$ over each element in J and eigenvalue $-\frac{\#J}{2}$ over ∞ . We thereby obtain the universal family of connections on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ having eigenvalues 0 and $\frac{1}{2}$ over each point in \underline{W} , where the $\frac{1}{2}$ -eigendirections over $i \in \underline{W} \setminus J$ and the 0-eigendirections over J are normalized to (20).

- **Twist:** Set $\kappa_0 = \kappa_1 = -\frac{1}{2}$ and $\kappa_i = \frac{1}{2}$ for all $i \in \underline{W} \setminus \{0, 1\}$. Apply positive elementary transformations in the parabolic directions corresponding to the $-\frac{1}{2}$ eigendirections of the universal family of connections (19). We obtain a new universal family of connections with eigenvalues 0 and $\frac{1}{2}$ over each $i \in \underline{W}$ such that the parabolic structure corresponding $\frac{1}{2}$ -eigendirections over each $i \in \underline{W}$ is normalized to

$$\begin{array}{cccccc} x=0 & x=1 & x=r & x=s & x=t & x=\infty \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} z_r \\ 1 \end{pmatrix} & \begin{pmatrix} z_s \\ 1 \end{pmatrix} & \begin{pmatrix} z_t \\ 1 \end{pmatrix} & \mathcal{O}_{\mathbb{P}^1}(-1) \end{array}.$$

- **Permutation :** For any $\sigma \in \mathfrak{S}(\{0, 1, r, s, t, \infty\})$, we obtain similar constructions for parabolic data normalized to

$$\begin{array}{cccccc} x=\sigma(0) & x=\sigma(1) & x=\sigma(r) & x=\sigma(s) & x=\sigma(t) & x=\infty \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} z_r \\ 1 \end{pmatrix} & \begin{pmatrix} z_s \\ 1 \end{pmatrix} & \begin{pmatrix} z_t \\ 1 \end{pmatrix} & \mathcal{O}_{\mathbb{P}^1}(-1) \end{array}$$

- **Galois involution:** Choose any of the above charts U together with its universal family of connections. First apply positive elementary transformations in all $\frac{1}{2}$ -eigendirections of the universal family of connections. We obtain logarithmic connections with eigenvalues 0 and $-\frac{1}{2}$ over each Weierstrass point. Then tensorize this connection by the unique logarithmic (rank 1) connection on $\mathcal{O}_{\mathbb{P}^1}(-3)$ having eigenvalues $\frac{1}{2}$ over each Weierstrass point. We obtain a new chart U' together with a universal connection such that $\Phi(U) = \Phi(U')$.

By construction, the above are indeed all affine charts of $\mathfrak{Con}(X/\iota)$. Moreover, the transition maps between charts are all birational.

Proposition 7.1. *The moduli space $\mathfrak{Con}(X/\iota)$ is entirely covered by the above charts, except for the preimages under Φ of the trivial connection on the trivial bundle E_0 and its twists.*

Proof. Firstly, note that all possible parabolic vector bundles $(\underline{E}, \underline{p})$ underlying a connection $\underline{\nabla}$ in $\text{Con}(X/\iota)$, where \underline{p} is given by the $\frac{1}{2}$ -eigendirections, occur in the above charts.

- If $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and \underline{p} is indecomposable, then at least three of the parabolics are not included in the total space of the destabilizing subbundle $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$ and are not included in the same $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$. Up to permutation, we can assume that this is the case for $\underline{p}_0, \underline{p}_1$ and \underline{p}_∞ . Any such configuration appears in the Canonical chart or a Switch. More precisely, we need to switch each parabolic contained in the destabilizing subbundle.
- If $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ and \underline{p} is decomposable, then we have two parabolics, which we can assume to be $\underline{p}_0, \underline{p}_1$ by permutation, defined by the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$ and the four others by some $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$. This configuration arises in the Twist chart.
- If $\underline{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ and \underline{p} is indecomposable, then at most one parabolic is included in the total space of the destabilizing subbundle and the Galois involution leads to an indecomposable parabolic configuration on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, which we have already treated.
- If $\underline{E} = \mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-3)$ and \underline{p} is decomposable, then all parabolics are defined by the total space of a same $\mathcal{O}_{\mathbb{P}^1}(-3) \hookrightarrow \underline{E}$. This configuration arises from

the Twist chart, namely when the 0-eigendirections over 0 and 1, as well as the $\frac{1}{2}$ -parabolics corresponding over $\underline{W} \setminus \{0, 1\}$ of some logarithmic connection on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ are included in the total space of a same subbundle $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

Secondly, we know from Section 3 that for every vector bundle E in $\mathfrak{Bun}^*(X)$, except for the trivial bundle and its twists, the space of ι -invariant $\mathfrak{sl}_2\mathbb{C}$ -connections on E is an affine \mathbb{C}^3 -space. Yet by construction, the universal connection we established for our charts provide a \mathbb{C}^3 -space of two-by-two non isomorphic connections on each parabolic bundle $(\underline{E}, \underline{p})$. The moduli space of irreducible or abelian connections on $E_0 \otimes L$ with $L^{\otimes 2} = \mathcal{O}_{X_t}$ in $\text{Con}(X)$, if we exclude the trivial connections, is only birationally isomorphic to \mathbb{C}^3 (see Section 3.3). The fact that we do indeed cover all of the mentioned connections follows from a more detailed but straightforward analysis. \square

7.2. The apparent map on $\mathfrak{Con}(X/\iota)$. Following [41], we will now recall the construction of the so-called *apparent map*, allowing us to prove Proposition 5.3. For a parabolic connection $(\underline{E}, \underline{p}, \nabla)$ defined on the main vector bundle $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, we can associate a morphism

$$\nabla \mapsto \varphi_{\nabla} \in H^0(\text{Hom}(\mathcal{O}_{\mathbb{P}^1}(-1), \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))) \simeq H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))$$

by composition of

$$\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E} \xrightarrow{\nabla} \underline{E} \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}) \rightarrow \mathcal{O}_{\mathbb{P}^1}(-2) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})$$

where the last arrow is just the projection on the second direct summand.

Remark 7.2. *Geometrically, the zeroes of the apparent map (which is an element of $H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3))$) are the coordinates of the (three) tangencies between the destabilizing section σ_{-1} of $\mathbb{P}(\underline{E})$ and the foliation on $\mathbb{P}(\underline{E})$ defined by flat sections of $\mathbb{P}(\nabla)$. On the other hand, these are precisely the positions of the apparent singular points appearing when we derive the associate 2nd order fuchsian equation from the “cyclic vector” $\mathcal{O}_{\mathbb{P}^1}(-1) \hookrightarrow \underline{E}$.*

We can extend the definition of the apparent map to so-called λ -connections

$$\nabla = \lambda \cdot \nabla_0 + c_r \Theta_r + c_s \Theta_s + c_t \Theta_t, \quad (\lambda, c_r, c_s, c_t) \in \mathbb{C}^4,$$

including Higgs fields (for $\lambda = 0$). There is a natural \mathbb{G}_m -action by multiplication on the moduli space of λ -connections so that a generic element ∇ , with $\lambda \neq 0$, is equivalent to a unique connection (in the usual sense), namely $\frac{1}{\lambda} \nabla$. After projectivization, we thus obtain a natural compactification of the moduli space of connections on \underline{E} (an affine 3-space) by the moduli space of projective Higgs fields (i.e. up to \mathbb{G}_m -action). In our coordinates, an element $(\lambda : c_r : c_s : c_t) \in \mathbb{P}^3$ denotes either a connection (when $\lambda \neq 0$) or a projective class of a Higgs field. It is proved in [41], Theorem 4.3, that the map $\nabla \mapsto \mathbb{P}\varphi_{\nabla}$, which is invariant under \mathbb{G}_m -action, defines an isomorphism from the moduli space of λ -connections up to \mathbb{G}_m -action onto $\mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))$. Moreover, we deduce a map

$$\text{Bun}_{\mu}^{ss}(X/\iota) \rightarrow \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^{\vee}$$

which to a parabolic bundle $(\underline{E}, \underline{p})$ associates the image under $\mathbb{P}\varphi$ of the hyperplane locus of Higgs bundles $\lambda = 0$. For $\frac{1}{6} < \mu < \frac{1}{4}$, this map is also an isomorphism.

On the other hand, looking at $\text{Bun}_\mu^{ss}(X/\iota)$ as extensions (see Section 5.3), we also get a natural isomorphism

$$\text{Bun}_\mu^{ss}(X/\iota) \xrightarrow{\sim} \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W}))^\vee$$

It follows from [41], proof of Theorem 4.3, that these two maps coincide.

Proof of Proposition 5.3. For $(R, S, T) \in \mathbb{C}^3$ finite, the corresponding parabolic bundle also belongs to $\text{Bun}_\mu^{ss}(X/\iota)$ and we can use the apparent map to compute the corresponding point $(b_0 : b_1 : b_2 : b_3) \in \mathbb{P}_b^3$. Precisely, the apparent map φ_{Θ_r} is given by the $(2, 1)$ -coefficient of Θ_r

$$\mathbb{P}\varphi_{\Theta_r} = \frac{R-1}{R-r}(x-r)(x-s)(x-t) \in \mathbb{P}H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1) \otimes \Omega_{\mathbb{P}^1}^1(\underline{W})) \simeq |\mathcal{O}_{\mathbb{P}^1}(3)|.$$

This provides a first equation

$$(R-r)b_0 - (\sigma_1 R - r(1+s+t))b_1 + (\sigma_2 R - r(s+t+st))b_2 - \sigma_3(R-1)b_3 = 0;$$

similar equations for Θ_s and Θ_t give the result. \square

7.3. A Lagrangian section of $\mathfrak{Con}(X) \rightarrow \mathfrak{Bun}(X)$. The rational section

$$\nabla_0 : \mathfrak{Bun}(X/\iota) \dashrightarrow \mathfrak{Con}(X/\iota)$$

constructed in Section 6.1 over the chart $\mathbb{P}_R^1 \times \mathbb{P}_S^1 \times \mathbb{P}_T^1$ is not invariant by the Galois involution of $\Phi : \mathfrak{Con}(X/\iota) \xrightarrow{2:1} \mathfrak{Con}(X)$, *i.e.* it defines a 2-section, but not a rational section $\mathfrak{Bun}(X) \dashrightarrow \mathfrak{Con}(X)$. One can easily deduce a rational section by taking the barycenter (recall that $\mathfrak{Con}(X) \rightarrow \mathfrak{Bun}(X)$ is an affine bundle) but it is not the simplest one. Here, we start back from the Tyurin parametrization of bundles to construct such an explicit section.

Like in Section 4, consider a generic data $(\underline{P}_1, \underline{P}_2, \lambda) \in X \times X \times \mathbb{P}^1$ and associate the parabolic structure $\tilde{\mathbf{p}}$ on $\tilde{E} := \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$ defined over

$$D := [\underline{P}_1] + [\iota(\underline{P}_1)] + [\underline{P}_2] + [\iota(\underline{P}_2)] \in |2K_X|,$$

by

$$(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)}) := \left(\lambda, -\lambda, \frac{1}{\lambda}, -\frac{1}{\lambda} \right)$$

(where λ_Q means the direction generated by $\lambda_Q e_1 + e_2$ over Q , for fixed independent sections e_1, e_2 over $X \setminus \{\infty\}$). After 4 elementary transformations, we get a bundle E with trivial determinant. A holomorphic connection ∇ on $E := \text{elm}_D^+(\tilde{E}, \tilde{\mathbf{p}})$ can be pulled-back to $\mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$ and we get a parabolic logarithmic connection $\tilde{\nabla}$ on this bundle with (apparent) singular points over D . In the basis $\langle e_1, e_2 \rangle$, we can write

$$\tilde{\nabla} : d + \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

where the trace is given by

$$\alpha + \delta = \frac{dx}{x-x_1} + \frac{dx}{x-x_2}$$

and the projective part takes the form (here z is the projective variable defined by $ze_1 + e_2$)

$$\mathbb{P}\tilde{\nabla} : dz - \gamma z^2 + (\alpha - \delta)z + \beta \quad \text{with} \quad \begin{cases} -\gamma &= \frac{A(x)}{(x-x_1)(x-x_2)} \frac{dx}{y} \\ \alpha - \delta &= \frac{by}{(x-x_1)(x-x_2)} \frac{dx}{y} \\ \beta &= \frac{C(x)}{(x-x_1)(x-x_2)} \frac{dx}{y} \end{cases}$$

where A, C are degree 3 polynomials in x and $b \in \mathbb{C}$. This is due to the fact that the connection has only simple poles over D and that it is invariant under the (normalized) lift of the hyperelliptic involution $h : (x, y, z) \mapsto (x, -y, -z)$. We note that e_1 and e_2 generate the two ι -invariant Tyurin subbundles. Moreover, these coefficients $\{(A, b, C)\}$ have to satisfy several additional conditions, namely the compatibility with the parabolic data, that eigenvalues are 0 and 1 (parabolic directed by 1) and the singularity is apparent, in the sense that it disappears after an elementary transformation in the parabolic. This gives 6 affine equations in the 9-dimensional space of coefficients $\{(A, b, C)\}$:

$$\begin{aligned} \text{parabolic data:} & \begin{cases} \lambda A(x_1) + by_1 + \frac{1}{\lambda} C(x_1) &= 0 \\ \frac{1}{\lambda} A(x_2) + by_2 + \lambda C(x_2) &= 0 \end{cases} \\ \text{eigenvalues:} & \begin{cases} 2\lambda A(x_1) + by_1 &= y_1(x_2 - x_1) \\ \frac{2}{\lambda} A(x_2) + by_2 &= y_2(x_1 - x_2) \end{cases} \\ \text{apparent:} & \begin{cases} 2y_1(\lambda A'(x_1) + \frac{1}{\lambda} C'(x_1)) + bF'(x_1) &= 0 \\ 2y_2(\frac{1}{\lambda} A'(x_2) + \lambda C'(x_2)) + bF'(x_2) &= 0 \end{cases} \end{aligned}$$

where $F(x) = x(x-1)(x-r)(x-s)(x-t)$. Viewing a Higgs field $\tilde{\Theta} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ as the difference of two connections, we get $\alpha + \delta = 0$ and, for the projective part, the corresponding linearized equations (with 0 right-hand-side). Starting with a connection $\tilde{\nabla}$ on $\tilde{E} = \mathcal{O}_X(-K_X) \oplus \mathcal{O}_X(-K_X)$ as above, *via* 4 elementary transformations, we get a holomorphic \mathfrak{sl}_2 -connection

$$(E, \nabla) = \text{elm}_D^+(\tilde{E}, \tilde{\nabla}, \tilde{\mathbf{p}})$$

on X whose parabolic data \mathbf{p} is supported by the strict transform of the line bundle $L_1 := \mathbb{C}\langle e_1 \rangle$ (we suppose $\lambda \notin \{0, \infty\}$ by genericity). Pushing it down, we get a parabolic connection on $X/\iota = \mathbb{P}_x^1$ for which L_1 becomes the destabilizing subbundle $\mathcal{O}_{\mathbb{P}^1}(-1)$.

Selecting the ι -invariant Tyurin subbundle L_1 , we have a natural generically finite map

$$X \times X \times \mathbb{P}_\lambda^1 \xrightarrow{16:1} \mathbb{P}_B^3$$

with Galois group generated by $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz} \rangle$ (see Section 4). Then, the Galois involution of $\mathbb{P}_B^3 \xrightarrow{2:1} \mathbb{P}_{\text{NR}}^3$ is induced by $\sigma_{1/z}$ which is permuting e_1 and e_2 . We can thus compute the apparent map of a connection $\tilde{\nabla}$ (or a Higgs field $\tilde{\Theta}$) with respect to e_1 and get that $\varphi_{\tilde{\nabla}} = A(x)$.

Remark 7.3. *The three zeroes of $A(x)$ define six points on X , which are the coordinates of the tangencies between e_1 and the foliation $\mathbb{P}(\tilde{\nabla})$ on $\mathbb{P}(\tilde{E})$.*

Like in the proof of Proposition 5.3 (see Section 7.2) we can use the apparent map for Higgs fields to compute the corresponding Bertram coordinates of \mathbb{P}_B^3 . A straightforward computation yields:

Proposition 7.4. *The natural map $X \times X \times \mathbb{P}_\lambda^1 \rightarrow \mathbb{P}_B^3$ is given by*

$$\begin{cases} b_0 &= \lambda y_2 - \frac{1}{\lambda} y_1 \\ b_1 &= \lambda x_1 y_2 - \frac{1}{\lambda} x_2 y_1 \\ b_2 &= \lambda x_1^2 y_2 - \frac{1}{\lambda} x_2^2 y_1 \\ b_3 &= \lambda x_1^3 y_2 - \frac{1}{\lambda} x_2^3 y_1 \end{cases}$$

It follows from [41] that a connection on a parabolic bundle belonging to the chart \mathbb{P}_B^3 is determined by its apparent map. It is particularly easy to see this fact in above equations: after prescribing $\varphi_{\tilde{\nabla}} = A(x) \in \mathbb{P}_A^3$ (up to homothecy), *i.e.* after prescribing the roots of $A(x)$, we get a unique solution (A, b, C) except when $A(x)$ lies in the hyperplane of Higgs bundles defined by Proposition 7.4 above. In the latter case, there is a solution (A, b, C) as a Higgs field which is unique up to an homothecy. Note that the group $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz} \rangle$ acts on connections (and Higgs fields) and the induced action on the coefficient $A(x)$ is by homothecy. It follows that the corresponding point $A(x) \in \mathbb{P}_A^3$ is invariant. The fourth involution $\sigma_{1/z}$ however permutes $A(x)$ and $C(x)$ (and changes the sign). In order to construct a rational section $\nabla_0 : \mathfrak{Bun}(X) \dashrightarrow \mathfrak{Con}(X)$, we can consider connections for which $A(x)$ and $C(x)$ are homothetic to each other, *i.e.* define the same point in \mathbb{P}_A^3 . A straightforward computation shows that there are exactly two possibilities:

$$\nabla^+ : b = \frac{\lambda^2 + 1}{\lambda^2 - 1}(x_1 - x_2) \quad \text{and} \quad A(x) = C(x) =$$

$$\begin{aligned} & \frac{1}{2(x_1 - x_2)^2} \frac{\lambda}{\lambda^2 - 1} \left((y_1 - y_2)(4x^3 - 6(x_1 + x_2)x^2 + 12x_1x_2x) - 6x_1x_2(x_2y_1 - x_1y_2) \right. \\ & \left. + 2(x_2^3y_1 - x_1^3y_2) - (x_1 - x_2)(x - x_1)(x - x_2) \left(y_1 \frac{F'(x_1)}{F(x_1)}(x - x_2) + y_2 \frac{F'(x_2)}{F(x_2)}(x - x_1) \right) \right) \end{aligned}$$

and

$$\nabla^- : b = \frac{\lambda^2 - 1}{\lambda^2 + 1}(x_1 - x_2) \quad \text{and} \quad A(x) = -C(x) =$$

$$\begin{aligned} & \frac{1}{2(x_1 - x_2)^2} \frac{\lambda}{\lambda^2 + 1} \left((y_1 + y_2)(4x^3 - 6(x_1 + x_2)x^2 + 12x_1x_2x) - 6x_1x_2(x_2y_1 + x_1y_2) \right. \\ & \left. + 2(x_2^3y_1 + x_1^3y_2) - (x_1 - x_2)(x - x_1)(x - x_2) \left(y_1 \frac{F'(x_1)}{F(x_1)}(x - x_2) - y_2 \frac{F'(x_2)}{F(x_2)}(x - x_1) \right) \right) \end{aligned}$$

This provides two “universal connections” over the parameter space $X \times X \times \mathbb{P}_\lambda^1$ which are each invariant under $\sigma_{1/z}$ and $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}^2 \rangle$, but permuted by σ_{iz} . Taking the barycenter of these two connections for each parameter $(\underline{P}_1, \underline{P}_2, \lambda)$ yields a fully invariant section

$$\nabla_0 := \frac{\nabla^+ + \nabla^-}{2}$$

whose coefficients are given by

$$\begin{aligned}
 \frac{b_0 y}{(x-x_1)(x-x_2)} &:= \frac{\lambda^4+1}{\lambda^4-1} \left(\frac{1}{x-x_1} - \frac{1}{x-x_2} \right) \\
 \frac{A_0(x)}{(x-x_1)(x-x_2)} &:= \frac{\lambda}{\lambda^4-1} \left\{ \left(\frac{y_2}{x-x_2} - \frac{\lambda^2 y_1}{x-x_1} \right) + \frac{(\lambda^2 y_1 - y_2)(2x-x_1-x_2)}{(x_1-x_2)^2} \right. \\
 &\quad \left. - \frac{\lambda^2 y_1 \frac{F'(x_1)}{F(x_1)}(x-x_2) + y_2 \frac{F'(x_2)}{F(x_2)}(x-x_1)}{2(x_1-x_2)} \right\} \\
 \frac{C_0(x)}{(x-x_1)(x-x_2)} &:= \frac{\lambda}{\lambda^4-1} \left\{ \left(\frac{\lambda^2 y_2}{x-x_2} - \frac{y_1}{x-x_1} \right) + \frac{(y_1 - \lambda^2 y_2)(2x-x_1-x_2)}{(x_1-x_2)^2} \right. \\
 &\quad \left. - \frac{y_1 \frac{F'(x_1)}{F(x_1)}(x-x_2) + \lambda^2 y_2 \frac{F'(x_2)}{F(x_2)}(x-x_1)}{2(x_1-x_2)} \right\}.
 \end{aligned} \tag{21}$$

Proposition 7.5. *The induced rational section*

$$\nabla_0 : \mathfrak{Bun}(X) \dashrightarrow \mathfrak{Con}(X)$$

is Lagrangian, and moreover regular over the open set of stable bundles.

Proof. This connection is well-defined provided that $\lambda^4 \neq 1$ and $x_2 \neq x_1$. We get a universal connection for all stable bundles. Indeed, we first check that all stable bundles off odd Gunning planes are covered by the open subset where the connection ∇_0 is well-defined:

$$X \times X \times \mathbb{P}_\lambda^1 \setminus (\{\lambda^4 = 1\} \cup \{x_1 = x_2\}) \rightarrow \mathbb{P}_{\text{NR}}^3 \setminus (\text{Kum}(X) \cup \Pi_{w_0} \cup \dots \cup \Pi_{w_\infty}).$$

We thus get a rational section $\nabla_0 : \mathfrak{Bun}(X) \dashrightarrow \mathfrak{Con}(X)$ which is holomorphic over stable bundles, off odd Gunning planes. We can check that it actually extends holomorphically along odd Gunning planes. It is sufficient to extend it outside intersections of odd Gunning planes since those form a codimension 2 subset. The Gunning plane Π_{w_0} comes from the indeterminacy locus $\{w_0\} \times X \times \{0\}$ of the map $X \times X \times \mathbb{P}_\lambda^1 \rightarrow \mathbb{P}_{\text{NR}}^3$. Precisely, a generic element of Π_{w_0} is obtained as follows. We first renormalize $z = w/\lambda$ so that parabolic directions become

$$(\lambda_{\underline{P}_1}, \lambda_{\iota(\underline{P}_1)}, \lambda_{\underline{P}_2}, \lambda_{\iota(\underline{P}_2)}) = (\lambda^2, -\lambda^2, 1, -1)$$

and then make the first two parabolic tending to 0 while $y_1 \rightarrow 0$ with some fixed slope $\frac{\lambda^2}{y_1} = c$. The limiting connection has now a double pole at w_0 , which disappears after two elementary transformations.

Finally, that this section is Lagrangian directly follows from straightforward verification. Precisely, following [41], in coordinates (\mathbf{a}, \mathbf{b}) defined by coefficients $\mathbf{a} = (a_0 : a_1 : a_2 : a_3)$ of $A_0(x)$ defined in (21) and Bertram coefficients $\mathbf{b} = (b_0 : b_1 : b_2 : b_3)$ defined in Proposition 7.4, the symplectic form is defined by

$$\omega = d\eta \quad \text{with} \quad \eta = \frac{a_0 db_0 + a_1 db_1 + a_2 db_2 + a_3 db_3}{a_0 b_0 + a_1 b_1 + a_2 b_2 + a_3 b_3}.$$

If we pull-back the 1-form η by the map

$$(\lambda, x_1, y_1, x_2, y_2) \mapsto (\mathbf{a}, \mathbf{b})$$

then we get the zero 1-form. □

Remark 7.6. *Over the open set of stable bundles, the natural map $\text{Con} \rightarrow \text{Bun}$ is a locally trivial Lagrangian fibration, which is also an affine \mathbb{A}^3 -fiber bundle. Over an affine open set, any affine bundle reduces to a vector bundle, namely its linear part T^*Bun^s . The existence of Lagrangian section (regular over the open set) shows that the reduction is actually symplectic with respect to the Liouville symplectic structure on T^*Bun^s .*

Remark 7.7. *There are precisely two Higgs fields invariant under $\sigma_{1/z}$:*

$$(x - x_1)(\lambda^2 z^2 - 1) \frac{dx}{y} \quad \text{and} \quad (x - x_2)(z^2 - \lambda^2) \frac{dx}{y}.$$

They are also permuted by σ_{iz} and invariant under $\langle \sigma_{12}, \sigma_\iota, \sigma_{iz}^2 \rangle$. We obtain a basis of the space of Higgs bundles by adding for example $\nabla^+ - \nabla^-$.

8. APPLICATION TO ISOMONODROMIC DEFORMATIONS

Our construction of the stack $\mathfrak{Con}^*(X)$ allows us to vary the parameter (r, s, t) defining the base curve

$$X : y^2 = x(x - 1)(x - r)(x - s)(x - t)$$

in

$$T := \{(r, s, t) \in \mathbb{C}^3 \mid r, s, t \neq 0, 1, r \neq s, r \neq t, s \neq t\}$$

in order to obtain a family $\mathcal{M} \rightarrow T$ such that $\mathcal{M}|_{(r,s,t)} = \mathfrak{Con}^*(X_{(r,s,t)})$. Roughly speaking, \mathcal{M} is the moduli space of triples (X, E, ∇) , where X is a curve of genus 2, E is a rank 2 vector bundle with trivial determinant bundle and ∇ a holomorphic trace free connection on E with either abelian (but non trivial) or irreducible monodromy. Locally in the variable $(r, s, t) \in T$, isomonodromic deformations are fibres of the monodromy map, defining an isomonodromic foliation \mathcal{F}_{iso} on \mathcal{M} . Note that an analytic family of connections over genus 2 curves with contractible parameter space is an isomonodromic deformation if and only if the connection is integrable and our isomonodromic foliation is thus defined by the integrability condition. Our aim in this section is to express explicitly this isomonodromic deformation, via the corresponding moduli space $\underline{\mathcal{M}} \rightarrow T$ such that $\underline{\mathcal{M}}|_{(r,s,t)} = \mathfrak{Con}^*(X_{(r,s,t)}/\iota)$. The integrability condition there is equivalent to a Garnier system. We then prove that the isomonodromic foliation is transversal to the locus of unstable bundles in \mathcal{M} and give a geometric interpretation of this result.

8.1. Darboux coordinates. We will use the notations of Section 7.1. The classical Darboux coordinates with respect to the symplectic form $\omega = dz_r \wedge dc_r + dz_s \wedge dc_s + dz_t \wedge dc_t$ on the Canonical chart U_0 are defined as follows. The vector $e_1 = {}^t(1, 0)$ becomes an eigenvector of the matrix connection for 3 points $x = q_1, q_2, q_3$ (counted with multiplicity), namely at the zeroes of the $(2, 1)$ -coefficient of the matrix connection:

$$(22) \quad -\rho + \sum_{i \in \{r,s,t\}} c_i \frac{(z_i - i)x - i(z_i - 1)}{x - i} = \left(-\rho + \sum_{i=1}^3 c_i(z_i - t_i) \right) \frac{\prod_{k=1}^3 (x - q_k)}{\prod_{i \in \{r,s,t\}} (x - i)}$$

At each of the three solutions $x = q_k$ of (22), the eigenvector $e_1 = {}^t(1, 0)$ is associated to the eigenvalue

$$(23) \quad p_k := -\frac{\rho}{q_k - 1} + \sum_{i \in \{r,s,t\}} c_i z_i \left(\frac{1}{q_k - 1} - \frac{1}{q_k - i} \right).$$

The equations (22) and (23) allow us to express our initial variables $(z_r, z_s, z_t, c_r, c_s, c_t)$ as rational functions of new variables $(q_1, q_2, q_3, p_1, p_2, p_3)$ as follows.

Set $\Delta = (q_1 - q_2)(q_2 - q_3)(q_3 - q_1)$ and

$$\Lambda = \rho + \sum_{\{k,l,m\}=\{1,2,3\}} \frac{p_k(q_k - r)(q_k - s)(q_k - t)}{(q_k - q_l)(q_k - q_m)}.$$

For $i \in \{r, s, t\}$, denote

$$\Lambda_i := \Lambda|_{i=1}$$

the rational function obtained by setting $i = 1$ in the expression of Λ . Then we have, for $\{i, j, k\} = \{r, s, t\}$

$$(24) \quad c_i = -\frac{(q_1 - i)(q_2 - i)(q_3 - i)}{i(i-1)(i-j)(i-k)}\Lambda \quad \text{and} \quad z_i = i\frac{\Lambda_i}{\Lambda}.$$

The rational map

$$(25) \quad \Pi : \mathbb{C}_{q,p}^6 \dashrightarrow \mathbb{C}_{z,c}^6 = U_0$$

has degree 6: the (birational) Galois group of this map is the permutation group on indices $k = 1, 2, 3$ for pairs (q_k, p_k) . In these new coordinates, the symplectic form writes

$$\omega = \sum_{i \in \{r,s,t\}} dz_i \wedge dc_i = \sum_{k=1}^3 dq_k \wedge dp_k.$$

We deduce a new atlas of \mathcal{M} with charts given locally in T by

$$(26) \quad \begin{array}{ccc} T \times \mathbb{C}_{q,p}^6 & \xrightarrow{\Pi} & T \times \mathbb{C}_{z,c}^6 \subset \underline{\mathcal{M}} \xrightarrow{\Phi} \mathcal{M} \\ ((r, s, t), q, p) & \xrightarrow{6:1} & ((r, s, t), z, c) \end{array}$$

of $\underline{\mathcal{M}}$ and \mathcal{M} respectively, each endowed again with a universal family of connections

$$(X_{r,s,t}, E_{r,s,t,z_1,z_2,z_3}, \nabla_{r,s,t,z_1,z_2,z_3,c_1,c_2,c_3}).$$

8.2. Hamiltonian system. For $i \in \{r, s, t\}$ define H_i by

$$i(i-1) \prod_{j \in \{r,s,t\} \setminus \{i\}} (j-i) \cdot H_i :=$$

$$\sum_{l=1}^3 \frac{\prod_{k \neq l} (q_k - i)}{\prod_{k \neq l} (q_k - q_l)} F(q_l) \left(p_l^2 - G(q_l)p_l + \frac{p_l}{q_l - i} \right) + \rho(\rho + \kappa_\infty) \prod_{l=1}^3 (q_l - i),$$

where $F(x) = x(x-1)(x-r)(x-s)(x-t)$ and $G(x) = \frac{\kappa_0}{x} + \frac{\kappa_1}{x-1} + \frac{\kappa_r}{x-r} + \frac{\kappa_s}{x-s} + \frac{\kappa_t}{x-t}$. Then, assuming $\kappa_i \notin \mathbb{Z}$ for any $i \in \{0, 1, r, s, t, \infty\}$, a local analytic map

$$\chi : (r, s, t) \mapsto (q_1, q_2, q_3, p_1, p_2, p_3)$$

induces an isomonodromic deformation of the connection (19) if, and only if,

$$(27) \quad \frac{\partial q_k}{\partial i} = \frac{\partial H_i}{\partial p_k} \quad \text{and} \quad \frac{\partial p_k}{\partial i} = -\frac{\partial H_i}{\partial q_k} \quad \forall i \in \{r, s, t\}, \quad k \in \{1, 2, 3\}.$$

In other words, the kernel of the 2-form

$$\Omega = \sum_{k=1}^3 dq_k \wedge dp_k + \sum_{i \in \{r,s,t\}} dH_i \wedge di,$$

defines a 3-dimensional (singular holomorphic) foliation \mathcal{F}_{iso} on \mathcal{M} which is transversal to $\{(r, s, t) = \text{const.}\}$ and χ parametrizes a leaf of this foliation. We will call it the isomonodromy foliation in the sequel. The tangent space to the foliation is defined by the 3 vector fields V_r, V_s, V_t given by

$$(28) \quad V_i := \frac{\partial}{\partial i} + \sum_{k=1}^3 \left(\frac{\partial H_i}{\partial p_k} \right) \frac{\partial}{\partial q_k} - \sum_{k=1}^3 \left(\frac{\partial H_i}{\partial q_k} \right) \frac{\partial}{\partial p_k}.$$

Note that the polar locus of these vector fields is given by $(q_1 - q_2)(q_2 - q_3)(q_1 - q_3) = 0$, namely the critical locus of the map (25). The induced (singular holomorphic) foliation on \mathcal{M} will be called the isomonodromy foliation \mathcal{F}_{iso} in the sequel.

8.3. Transversality to the locus of Gunning bundles.

Theorem 8.1. *For any even theta-characteristic ϑ , the locus $\{(X, E_\vartheta, \nabla) \in \mathcal{M}\}$ of connections on the Gunning bundle E_ϑ , is transversal to the isomonodromy foliation \mathcal{F}_{iso} .*

Proof. Up to permutation, we can assume

$$\vartheta = \mathcal{O}([w_{t_1}] + [w_{t_2}] - [w_{t_3}]) = \mathcal{O}([w_0] + [w_1] - [w_\infty]).$$

According to Section 3.5, the two pre-images under Φ of E_ϑ are $(\underline{E}, \underline{p})$ with $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-2) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, where

- The parabolics \underline{p}_i are given by the total space of the fibres of $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$ for all $i \in \{r, s, t\}$. Moreover, there is a line subbundle $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$ such that \underline{p}_0 and \underline{p}_∞ are given by the corresponding fibres, and \underline{p}_1 lies on neither of these two line subbundles.
- The parabolics \underline{p}_i are given by the total space of the fibres of $\mathcal{O}_{\mathbb{P}^1}(-1) \subset \underline{E}$ for all $i \in \{0, 1, \infty\}$. Moreover, there is a line subbundle $\mathcal{O}_{\mathbb{P}^1}(-2) \hookrightarrow \underline{E}$ such that \underline{p}_r and \underline{p}_t are given by the corresponding fibres, and \underline{p}_s lies on neither of these two line subbundles.

For any holomorphic connection on E_ϑ , the parabolic structure on $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$ can be normalized to one of the two above. Up to the permutation $\sigma = \begin{pmatrix} 0 & 1 & r & s & t & \infty \\ r & s & 0 & 1 & \infty & t \end{pmatrix}$, it is enough to consider the first configuration. Note that after a Möbius-transformation in the basis, the configuration after permutation corresponds to the configuration before permutation, but for different values of $(r, s, t) \in T$. We are led to the parabolic structure

$$\begin{array}{cccccc} x=0 & x=1 & x=r & x=s & x=t & x=\infty \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \mathcal{O}_{\mathbb{P}^1}(-1) \end{array}$$

on $\underline{E} = \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$, in other words $(z_r, z_s, z_t) = (\infty, \infty, \infty)$. This parabolic structure is non visible in the canonical chart U_0 , the Twisted chart and Switched charts for $J \neq \{r, s, t\}$. Moreover, we can discard Permuted charts by the argument given above and Galois involution charts because we do not need them to cover \mathcal{M} . Therefore, the only chart containing the Gunning bundle E_ϑ we need to consider is the Switched chart for $J = \{r, s, t\}$, i.e. we consider the Darboux chart with respect to the parabolic

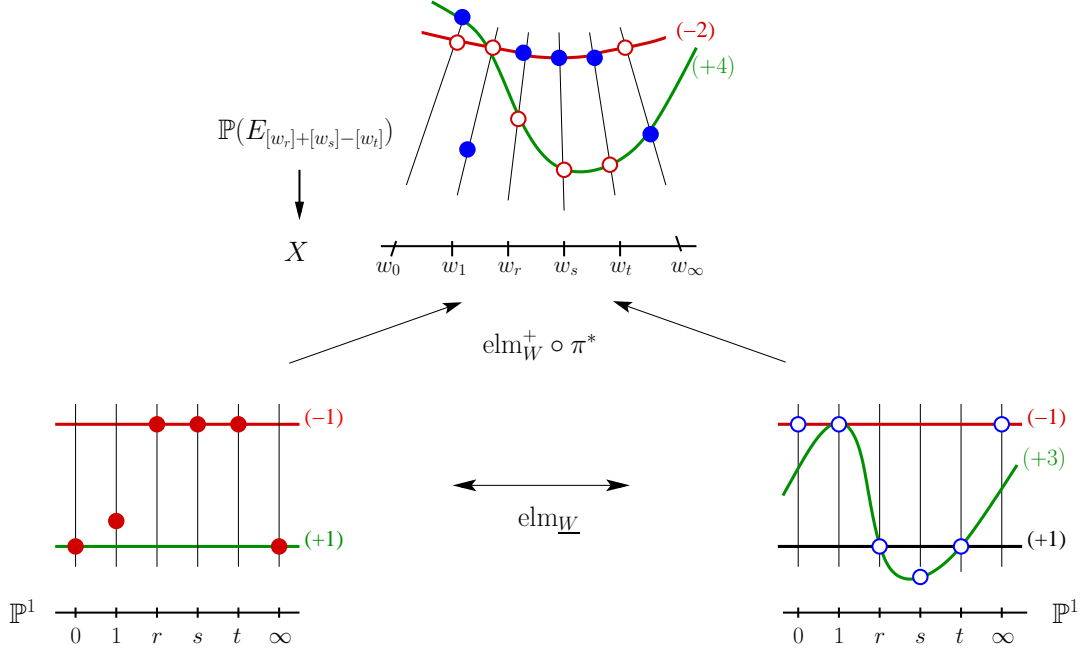


FIGURE 15. The two parabolic bundles corresponding to E_ϑ with $\vartheta = [w_r] + [w_s] - [w_t]$ under the lifting map Φ .

structure attached to the 0-eigenvalue over $x = r, s, t$. Consider now spectral data (20) with

$$\kappa_0 = \kappa_1 = \kappa_\infty = \frac{1}{2} \quad \text{and} \quad \kappa_r = \kappa_s = \kappa_t = -\frac{1}{2} \quad (\Rightarrow \rho = \frac{1}{2}).$$

Connections on our Gunning bundle E_ϑ then correspond to those parabolic connections with $z_r, z_s, z_t \in \mathbb{C}$ (finite) but having e_1 as eigenvector over $x = r, s, t$. This means that $c_r = c_s = c_t = 0$ and we are just looking at the family Σ defined by ∇_0 when the parabolic data $z = (z_r, z_s, z_t) \in \mathbb{C}_z^3$ is arbitrary. We want to prove that the isomonodromic foliation is transversal to

$$\Sigma := \{((r, s, t), (z, c)) \in T \times \mathbb{C}^6 \mid c_i = 0\}.$$

Locally in T , the linear subspace $\Sigma \in \mathbb{C}_{r,s,t,z,c}^9$ is the image of the linear subspace $\Sigma^{\text{Darb}} \in \mathbb{C}_{r,s,t,q,p}^9$ defined by

$$\Sigma^{\text{Darb}} = \{q_1 = r, q_2 = s, q_3 = t\}.$$

Precisely, the map $\Pi : \mathbb{C}_{r,s,t,q,p}^9 \dashrightarrow \mathbb{C}_{r,s,t,z,c}^9$ induces an affine transformation

$$\Sigma^{\text{Darb}} \rightarrow \Sigma ; (r, s, t, p) \mapsto (r, s, t, z)$$

with

$$z_r = r(2(r-1)p_1 + 1), \quad z_s = s(2(s-1)p_2 + 1), \quad z_t = t(2(t-1)p_3 + 1).$$

Lemma 8.2. *There is a neighborhood of Σ^{Darb} such that Π restricted to this neighborhood is a local diffeomorphism. Moreover, Σ^{Darb} is sent surjectively onto Σ .*

Proof. We can check by direct computation from (22) and (23) that

- the c_i 's have poles only at $q_k = q_l$, thus far from Σ^{Darb} ,
- the z_i 's have poles also not intersecting Σ^{Darb} ,

- the q_k 's can be inversely defined near Σ ,
- the p_k 's are then regular near Σ .

For this last fact, we have to take special care with the indefinite terms $\frac{c_r z_r}{q_1 - r}$, $\frac{c_s z_s}{q_2 - s}$, $\frac{c_t z_t}{q_3 - t}$, occuring in formula (23). For instance, for p_1 , we can use the (unitary) equation for the q_i 's given by formula (22) and then substitute

$$\frac{c_r z_r}{q_1 - r} = -\frac{c_r z_r (q_2 - r)(q_3 - r)}{Q(r)}$$

with $Q(x) = (x - q_1)(x - q_2)(x - q_3)$, leading to

$$p_1 = -\frac{\rho}{q_1 - 1} + \sum_{i \in \{r, s, t\}} \frac{c_i z_i}{q_1 - 1} + \frac{z_r (q_2 - r)(q_3 - r)(\rho - \sum_{i \in \{r, s, t\}} c_i (z_i - i))}{r(r - 1)(r - s)(r - t)} - \frac{c_s z_s}{q_1 - s} - \frac{c_t z_t}{q_1 - t}.$$

The right-hand-side is now well-defined and analytic in a neighborhood of Σ . \square

We want to show that the isomonodromy foliation \mathcal{F}_{iso} is transversal to Σ . By the previous lemma it is enough to prove the transversality of Σ^{Darb} with the vector fields V_i defined in (28). Modulo the vector fields

$$\frac{\partial}{\partial r} + \frac{\partial}{\partial q_1}, \quad \frac{\partial}{\partial s} + \frac{\partial}{\partial q_2}, \quad \frac{\partial}{\partial t} + \frac{\partial}{\partial q_3}$$

and

$$\frac{\partial}{\partial p_1}, \quad \frac{\partial}{\partial p_2}, \quad \frac{\partial}{\partial p_3},$$

that are tangent to Σ^{Darb} , the vector fields V_i are equivalent to

$$\begin{aligned} \tilde{V}_r &= -\frac{\partial}{\partial q_1} + \sum_{k=1}^3 \left(\frac{\partial H_r}{\partial p_k} \right) \frac{\partial}{\partial q_k} \\ \tilde{V}_s &= -\frac{\partial}{\partial q_2} + \sum_{k=1}^3 \left(\frac{\partial H_s}{\partial p_k} \right) \frac{\partial}{\partial q_k} \\ \tilde{V}_t &= -\frac{\partial}{\partial q_3} + \sum_{k=1}^3 \left(\frac{\partial H_t}{\partial p_k} \right) \frac{\partial}{\partial q_k} \end{aligned}$$

But the corresponding matrix writes (with I denoting the identity 3-by-3 matrix)

$$(\tilde{V}_r, \tilde{V}_s, \tilde{V}_t) = \left(\frac{\partial H_i}{\partial p_k} \right)_{i \in \{r, s, t\}, k \in \{1, 2, 3\}} - I$$

and we obtain

$$(\tilde{V}_1, \tilde{V}_2, \tilde{V}_3)|_{(q_1, q_2, q_3) = (r, s, t)} = \frac{1}{2} \cdot I,$$

which is clearly invertible. This proves transversality of the isomonodromic foliation \mathcal{F}_{iso} with the locus Σ of our even Gunning bundle in $\underline{\mathcal{M}}$. The transversality of \mathcal{F}_{iso} with the locus $\Phi(\Sigma)$ of our even Gunning bundle in \mathcal{M} then follows from the fact that the Gunning bundle is not in the ramification locus of the action of the Galois involution in the considered Switched chart : the two-fold cover $(r, s, t, z, c) \xrightarrow{2:1} \Phi(r, s, t, z, c)$ is a local diffeomorphism in a neighborhood of Σ . \square

8.4. Projective structures and Hejhal's theorem.

8.4.1. *Projective structures.* A projective structure on the Riemann surface X is the data of an atlas of charts $f_i : U_i \rightarrow \mathbb{P}^1$ (holomorphic diffeomorphisms) such that transition charts $\varphi_{ij} := f_j \circ f_i^{-1}$ are Moebius transformations in restriction to their set of definition: $\varphi_{ij} \in \mathrm{PGL}_2(\mathbb{C})$. Two projective atlases define the same projective structure if their union (concatenation) also forms a projective atlas. This notion goes back to the works of Schwarz on the hypergeometric equation where the projective charts are locally defined as quotients of independant solutions of a given 2nd-order differential equation $u'' + f(x)u' + g(x)u = 0$; equivalently, after normalization $u'' + \frac{\phi(x)}{2}u = 0$, $\phi = g - \frac{f'}{2} - \frac{f^2}{4}$, local projective charts are solutions of the differential equation $\{f, x\} = \phi$ where $\{f, x\} = \left(\frac{f''}{f'}\right)' - \frac{1}{2}\left(\frac{f''}{f'}\right)^2$ is the Schwarzian derivative with respect to x (see [53, chapter VIII] for this point of view). In the hypergeometric case, the projective structure has singular points at poles of the differential equation. In our case, we can define a (non singular) projective structure on the curve

$$X_{(r,s,t)} : \{y^2 = F(x)\}, \quad F(x) = x(x-1)(x-r)(x-s)(x-t)$$

by a unique differential equation of the form

$$u'' + \left(\frac{1}{2} \frac{F'}{F}\right) u' + \left(\frac{x^3 + b_2x^2 + b_1x + b_0}{2F}\right)$$

(where u' and F' mean partial derivative with respect to x). When we let the complex structure (r, s, t) of the curve vary in $T \subset \mathbb{C}^3$, where

$$T = \{(r, s, t) \in \mathbb{C}^3 \mid r, s, t \neq 0, 1, r \neq s, r \neq t, s \neq t\},$$

the space of projective structures identifies with

$$T \times \mathbb{C}_b^3,$$

where $\mathbb{C}_b^3 = \{(b_0, b_1, b_2) \in \mathbb{C}^3\}$. Following [29], the data of a global non singular projective structure on X is also equivalent to the data of a SL_2 -connection on a Gunning bundle (E_ϑ, ∇) . In fact, up to the choice of E_ϑ , we have a one-to-one correspondance between connections ∇ on E_ϑ and projective structures on X .

The monodromy of a projective structure is by definition the monodromy of the connection ∇ , of the 2nd-order differential equation, or of any local projective chart (that can be analytically continued along any loop). After lifting to the Teichmüller space, namely the universal cover $\tilde{T} \rightarrow T$, the monodromy map

$$\mathrm{Mon} : \tilde{T} \times \mathbb{C}_b^3 \rightarrow \mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2)/\mathrm{PGL}_2.$$

is well-defined and analytic.

8.4.2. *Hejhal's theorem.* A problem which goes back to the work of Poincaré on Fuchsian functions was to decide which kind of representation $\mathrm{Hom}(\pi_1(X, w), \mathrm{SL}_2)/\mathrm{PGL}_2$ arise as the monodromy of a projective structure, i.e. as monodromy of (E_ϑ, ∇) , maybe deforming the complex structure of X . Counting dimensions, we get 3 parameters (r, s, t) for the curve and then 3 other parameters $b = (b_0, b_1, b_2)$ for the projective structure on the curve. Since the dimension of representations space is 6, one expect to realize most of them as monodromy. This was indeed proved in [21]: a representation can be realized if, and only if, it is not conjugated to a unitary representation, and it has Zariski dense image in $\mathrm{SL}_2(\mathbb{C})$. Some time earlier, D. A. Hejhal proved in [30] a local version:

Theorem 8.3 (Hejhal). *The monodromy map*

$$\text{Mon} : \tilde{T} \times \mathbb{C}_b^3 \rightarrow \text{Hom}(\pi_1(X, w), \text{SL}_2)/\text{PGL}_2$$

is a local diffeomorphism.

Going back to the isomonodromy point of view, consider the Gunning bundle E_ϑ with $\vartheta = \mathcal{O}_X([w_0] + [w_1] - [w_\infty])$. The locus of projective structures is given by the subspace Σ of those triples (X, E, ∇) with $E = E_\theta$ in the total moduli stack \mathcal{M} . The leaves of the isomonodromy foliation are locally defined as the fibres of the monodromy map RH . That the monodromy map $\text{RH}|_\Sigma$ restricted to the locus Σ of projectives structures is a local diffeomorphism is therefore equivalent to saying that the isomonodromic foliation is transversal to Σ . With 8.1, we have therefore provided a new proof of Hejhal's theorem.

Remark 8.4. *The topological transversality of Σ with the isomonodromy leaves, or equivalently the openness of the monodromy map, also follows from the main result in [31]. Indeed, the projective structure induced on X by taking the cyclic vector $\mathcal{O}_{\mathbb{P}^1}$ has no apparent singular point (since all $q_i = t_i$) and cannot be deformed isomonodromically (see [31, section 1.2]).*

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